PAPER CODE NO.
MA06

THE UNIVERSITY
of LIVERPOOL

SUMMER 1998 EXAMINATIONS

Degree of Bachelor of Science : Year 0
Degree of Bachelor of Science : Year 1
Degree of Bachelor of Engineering : Year 0

VECTORS AND KINEMATICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.

SECTION A

1. The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, with respect to an origin O. Express each of the following in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

   (a) $\overrightarrow{BC}$;

   (b) the position vector with respect to O of the point D such that $ABDC$ is a parallelogram;

   (c) the position vector with respect to O of the point E such that $\overrightarrow{BE} = 3\overrightarrow{BA}$.

   [**8 marks**]

2. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be mutually orthogonal unit vectors. Suppose that

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \qquad \text{and} \qquad \mathbf{v} = \mathbf{i} - 4\mathbf{j} - \mathbf{k}.$$

   Find

   (a) the lengths of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$;

   (b) a unit vector parallel to $\mathbf{u}+\mathbf{v}$;

   (c) $\mathbf{u}.\mathbf{v}$;

   (d) a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   [**11 marks**]

3. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors.

   (a) What can you deduce from the statement $\mathbf{a}.\mathbf{b} = 0$?

   (b) What can you deduce from the statement $\mathbf{b} \times \mathbf{c} = \mathbf{0}$?

   (c) What can you deduce from the statement $(\mathbf{a} \times \mathbf{b}).\mathbf{c} = 0$.

   [**7 marks**]

4. The points A, B and C have Cartesian coordinates $(1,-3,4)$, $(2,-4,4)$ and $(2,-2,2)$, respectively. Find

   (a) the lengths of the sides of triangle ABC;

   (b) the angles of the triangle ABC;

   (c) the area of triangle AOB, where O is the origin.

[**12 marks**]

5. The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to the coordinate axes Ox, Oy and Oz respectively. The points A and B have coordinates $(1,-1,1)$ and $(2,1,-1)$ respectively.

   (a) Express $\overrightarrow{AB}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

   (b) Find the vector equation of the line through A and B.

   (c) Find the Cartesian equation of the plane through A and perpendicular to AB.

[**9 marks**]

6. The equation of motion of a particle of mass m, moving under a constant force $\mathbf{F}$, is

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{F}.$$

The particle is projected from the origin at time $t = 0$ with velocity $\mathbf{u}$. Integrate the equation of motion twice to obtain $\mathbf{r}$ as a function of t.

Suppose now that $\mathbf{F} = -mg\mathbf{k}$ and $\mathbf{u} = v(\mathbf{i} + \mathbf{k})$, where $\mathbf{i}$ is a unit vector in a horizontal direction and $\mathbf{k}$ is a unit vector in the vertically upwards direction, and g and v are constants. Find the time at which the particle reaches the highest point of its trajectory.

[**8 marks**]

SECTION B

7. The position vectors with respect to O of the vertices A and B of the triangle OAB are $\mathbf{a}$ and $\mathbf{b}$ respectively. The mid point of AB is D and that of OA is E.

   (a) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the vector equations of the line l_1 through B and$E$ and the line l_2 through O and D.

   (b) Let G be the point of intersection of l_1 and l_2. Find $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (c) Write down the equation of the line l_3 through A and G.

   (d) Show that l_3 passes through the mid point of OB.

[15 marks]

8. The planes Π_1 and Π_2 have equations

$$x + 2y - z = 4 \qquad \text{and} \qquad 2x - y - 3z = 3,$$

respectively, with respect to Cartesian axes $Oxyz$. Find

   (a) the angle between the planes Π_1 and Π_2;

   (b) the distance of the origin O from Π_1;

   (c) the equation of the line l of intersection of the planes Π_1 and Π_2 in terms of a parameter λ;

   (d) the coordinates of two (different) points on the line l.

[15 marks]

9. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be mutually orthogonal unit vectors. Suppose that

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \qquad \mathbf{b} = \mathbf{j} + 3\mathbf{k} \qquad \text{and} \qquad \mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

(a) (i) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.

(ii) Express $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(b) Establish whether the lines with vector equations

$$\mathbf{r} = \mathbf{a} + 2\mathbf{c} + \lambda\mathbf{b} \qquad \text{and} \qquad \mathbf{r} = \mathbf{c} + \mu(\mathbf{a} - \mathbf{c})$$

do or do not intresect, giving reasons.

(c) Show that the position vector of the point on the line with equation

$$\mathbf{r} = \mathbf{c} + \mu(\mathbf{a} - \mathbf{c})$$

which is nearest to the origin can be written as

$$\mathbf{c} + \frac{\mathbf{c} \cdot (\mathbf{c} - ab)}{|\mathbf{a} - cb|^2}(\mathbf{a} - cb)$$

(*Hint:* consider the distance squared $\mathbf{r} \cdot \mathbf{r}$).

[**15 marks**]

10. (a) The position vector, with respect to a fixed origin O, of a particle at time t is

$$\mathbf{r} = (a\cos\omega t)\mathbf{i} + (a\sin\omega t)\mathbf{j},$$

where $\mathbf{i}$ and $\mathbf{j}$ are fixed mutually orthogonal unit vectors and a and ω are constants.

(i) Show that the path of the particle is a circle centre O.

(ii) Find the velocity $\mathbf{v}$ of the particle at time t and deduce that $\mathbf{r} \times \mathbf{v}$ is constant.

(iii) Show that the acceleration of the particle is, at all times, of the form $\alpha\mathbf{r}$ and give the value of the constant α.

(b) An aircraft has constant velocity $200(\mathbf{i} - 2\mathbf{j})$ km h^{-1} with respect to the wind, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors pointing due East and due North, respectively. The wind is blowing *from* the South-West at a speed of $50\sqrt{2}$ km h^{-1} with respect to the ground. What is the speed of the aircraft with respect to the ground.

[**15 marks**]