TIME ALLOWED: Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B. The total of the marks available on Section A is 55.

SECTION A

1. The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, with respect to an origin O. The point D is the mid point of AB and X is a point such that $ADXC$ is a parallelogram. Express $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OX}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

[8 marks]

2. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be mutually orthogonal unit vectors. Suppose that $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{k}$. Find

   (a) the lengths of $\mathbf{a}$ and $\mathbf{b}$;

   (b) the angle between $\mathbf{a}$ and $\mathbf{b}$, to the nearest degree;

   (c) a unit vector parallel to $\mathbf{a} - \mathbf{b}$.

   Assume now that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a right handed set. Find also

   (d) $\mathbf{a} \times \mathbf{b}$.

[**12 marks**]

3. Let $\mathbf{u}$ be a unit vector. Show that, for all vectors $\mathbf{a}$, the vector $\mathbf{a} - (\mathbf{a}.\mathbf{u})\mathbf{u}$ is orthogonal to $\mathbf{u}$.

[**3 marks**]

4. The points A, B and C have Cartesian coordinates $(1, 2, 1)$, $(3, 4, 2)$ and $(-3, 1, 2)$, respectively. Find

   (a) the length of AC;

   (b) the area of the triangle ABC;

   (c) the volume of the parallelepiped with edges parallel to $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

[**11 marks**]

5. Points A, B, and C have position vectors, with respect to an origin O, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. Write down the vector equation for the line through A and parallel to a given vector $\mathbf{s}$.

   Find the vector equation for the line l through A and the mid point of BC.

[**5 marks**]

6. Let P be the point with Cartesian coordinates $(1, 2, 3)$. Find the Cartesian equation of the plane through the origin O and orthogonal to OP.

[4 marks]

7. Suppose that $\mathbf{a}.\mathbf{b} \times \mathbf{c} = \alpha$. What is the value of $(\mathbf{b} - \mathbf{c}).\mathbf{a} \times \mathbf{c}$?

[4 marks]

8. The equation of motion of a particle of mass m, moving under a constant gravitational force $-mg\mathbf{k}$, where $\mathbf{k}$ is a unit vector directed vertically upwards, is

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -g\mathbf{k}\,.$$

The particle is thrown with velocity $\mathbf{u}$ from a height h above the ground. Choose a suitable origin and integrate the equation of motion twice to obtain $\mathbf{r}$.

Assume that $\mathbf{u}.\mathbf{k} > 0$. What is the greatest height above the ground reached by the particle?

[8 marks]

S E C T I O N   B

9. The planes Π_1 and Π_2 have equations

$$4x + y + z = 4 \qquad \text{and} \qquad 7x + y - 2z = 10\,,$$

respectively, with respect to Cartesian axes $Oxyz$.

(a) Find the angle between the planes Π_1 and Π_2.

(b) Find the line of intersection of the planes Π_1 and Π_2 in terms of a parameter λ.

(c) Find the distance of the point P with coordinates $(4, -3, 3)$ from the plane Π_1 and determine whether or not P lies on the same side of Π_1 as the origin.

(d) Find the equation of the plane Π_3 through P and at right angles to both Π_1 and Π_2.

[15 marks]

10. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be mutually orthogonal unit vectors.

(a) Suppose that $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

i. Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent.

ii. Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ are linearly dependent.

iii. Express $\mathbf{d}$ as a linear combination of $\mathbf{a}$ and $\mathbf{b}$

(b) Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{k}$ and $\mathbf{w} = -\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

i. Verify that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal.

ii. Suppose that $\mathbf{s} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$. Show that $\alpha = \mathbf{s}.\mathbf{u}/|\mathbf{u}|^2$ and write down similar expressions for β and γ.

iii. Express $\mathbf{i}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

[15 marks]

11. (a) A ship sailing due north at a speed of 20 km h^{-1} observes another ship 2 km to the north east which appears, to those on the first ship, to be travelling north-west at 15 km h^{-1}. Find the true speed and direction of motion of the second ship.

(b) Find the distance between the lines

$$\mathbf{r} = \mathbf{i} + \lambda(2\mathbf{j} - 5\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = \mathbf{j} + \mu(3\mathbf{i} + 2\mathbf{k}) .$$

[**15 marks**]

12. (a) The position vector at time t of a particle P, with respect to a fixed origin O, is

$$\mathbf{r}(t) = a(\omega t - \sin \omega t)\mathbf{i} + a(1 - \cos \omega t)\mathbf{j},$$

where a and ω are constants and $\mathbf{i}$ and $\mathbf{j}$ are constant, mutually orthogonal unit vectors. Find

i. the velocity of the particle at time t;

ii. the acceleration of the particle at time t.

Show that the magnitude of the acceleration is constant,

What is the direction of motion when the speed is greatest?

(b) The position vector, with respect to a fixed origin O, of a particle at time t is $\mathbf{r} = \mathbf{r}(t)$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r} \times \ddot{\mathbf{r}}\,.$$

The equation of motion of the particle is

$$\ddot{\mathbf{r}} = \mathbf{F}\,.$$

Show that, if $\mathbf{F}$ is always parallel to $\mathbf{r}$, then $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$ is a constant vector.

Deduce that the motion of the particle lies in the plane $\mathbf{r}.\mathbf{h} = 0$.

Show also that, if $|\mathbf{r}|$ is constant, then $\mathbf{r}$, $\dot{\mathbf{r}}$ and $\mathbf{h}$ are mutually orthogonal vectors.

[15 marks]