1 (a) $\dfrac{x^5(y^2z^3)^2}{x^{-2}y^4z^5} = \dfrac{x^5y^4z^6}{x^{-2}y^4z^5} = x^7z$ [2 marks]

(b) $\dfrac{b^2-9}{b^2-6b+9} = \dfrac{(b-3)(b+3)}{(b-3)^2} = \dfrac{b+3}{b-3}$. [2 marks]

2 $\dfrac{3}{x^2+3x} + \dfrac{1}{x+3} = \dfrac{3+x}{x(x+3)} = \dfrac{1}{x}$ [4 marks]

3 (a) $x^2 - x - 20 = (x-5)(x+4)$ so solutions are $x = 5, x = -4$. [2 marks]

(b) Using the quadratic formula $x = \dfrac{13 \pm \sqrt{169 - 4 \times 10 \times (-3)}}{6} = \dfrac{13 \pm 17}{6} = 5$ or $-\dfrac{2}{3}$ [2 marks]

4 (a) $y = 3x - 6$ represents a straight line with slope 3 meeting the y-axis at $y = -6$. [2 marks]

(b) $y = x^2 + 4x - 5$ is a quadratic curve, which is U-shaped, crossing the y-axis at $y = -5$ and the x-axis at $x = 1, x = -5$. The curve is symmetric about the line $x = -4/2 = -2$. The bottom point is at $x = -2, y = -9$. [3 marks]

(c) $y = |x^2 + 4x - 5|$ is given from (b) by reflecting the part below the x-axis in the x-axis. [3 marks]

5 Put $y = \dfrac{1+3x}{2x-5}$, and solve for x in terms of $y = f(x)$. Then $(2x-5)y = 1 + 3x$ so $2xy - 5y = 1 + 3x$, giving $x(2y-3) = 1 + 5y$.

Thus $x = f^{-1}(y) = \dfrac{1+5y}{2y-3}$ and so $f^{-1}(x) = \dfrac{1+5x}{2x-3}$. [3 marks]

6 (a) Either use the formula $a\dfrac{1-r^n}{1-r}$ with $r = 3, a = 3, n = 6$, giving $3 \times \dfrac{3^6-1}{3-1} = 1092$, or simply add up the 6 terms. [3 marks]

(b) The formula is $\dfrac{a}{1-r}$. [1 marks]

Here $a = \frac{5}{8}, r = \frac{5}{8}$, giving the sum as $\dfrac{\frac{5}{8}}{1-\frac{5}{8}} = \dfrac{\frac{5}{8}}{\frac{3}{8}} = \frac{5}{3}$. [2 marks]

7 (a) $\displaystyle\lim_{n\to\infty} \frac{2-3n+5n^2}{5-2n^2} = \lim_{n\to\infty} \frac{\frac{2}{n^2} - \frac{3}{n} + 5}{\frac{5}{n^2} - 2} \to \frac{5}{-2} = -\frac{5}{2}$ as $n \to \infty$. [2 marks]

(b) Putting $x = -2$ in the bottom of the fraction gives 0, as also in the top. Factorise to write

$$\frac{x^2-x-6}{x^2-4} = \frac{(x-3)(x+2)}{(x-2)(x+2)} = \frac{x-3}{x-2}$$

Now put $x = -2$ to get the limit $\frac{5}{4}$. [2 marks]

8 (a) Put $u = 3x - 5$. Then $y = (3x-5)^5 = u^5$ so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4 \times 3 = 15(3x-5)^4.$$

[2 marks]

(b) Put $u = x^3 + 2$. Then $y = (x^3+2)^{\frac{4}{3}} = u^{\frac{4}{3}}$ and

$$\frac{dy}{dx} = \tfrac{4}{3}u^{\frac{1}{3}} \times \frac{du}{dx} = \tfrac{4}{3}u^{\frac{1}{3}} \times 3x^2 = 4x^2(x^3+2)^{\frac{1}{3}}.$$

[3 marks]

(c) $y = x^6 \sin x = uv$ with $u = x^6, v = \sin x$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^6 \cos x + 6x^5 \sin x.$$

[3 marks]

9 The slope of the tangent is the value of $\frac{dy}{dx}$ at $x = 1$. Now $\frac{dy}{dx} = 6x^2$, so the slope is 6. When $x = 1$ we have $y = -3$, so the tangent line has equation $y + 3 = 6(x-1)$, giving $y = 6x - 9$. [3 marks]

10 (a) $\int (\cos x + 3x^4 - 2)\, dx = \int \cos x\, dx + 3\int x^4\, dx - 2\int 1\, dx = \sin x + \frac{3}{5}x^5 - 2x + C$

[4 marks]

(b) $\int e^{6x}\, dx = \frac{1}{6}e^{6x} + C.$ [2 marks]

11 (a) $\int_0^{\frac{\pi}{6}} \sin 6x\, dx = \left[-\frac{1}{6}\cos 6x\right]_0^{\frac{\pi}{6}} = -\frac{1}{6}\cos\pi + \frac{1}{6}\cos 0 = \frac{1}{3}.$ [3 marks]

(b) Substitute $u = 4x - 3$ so that $du = 4dx$. Then

$$I = \int_1^2 \frac{4}{4x-3}\, dx = \int_{x=1}^{x=2} \frac{1}{u}\, du = \int_{u=1}^{u=5} \frac{1}{u}\, du = [\ln u]_1^5 = \ln 5 - \ln 1 = \ln 5 (= 1.61).$$

[3 marks]

12 (i) Differentiate the LHS to get

$$3 \times 4x^3 + 2(x^2\frac{dy}{dx} + 2xy) - 3y^2\frac{dy}{dx}$$

[4 marks]

The RHS has derivative 0 giving the equation $12x^3 + 2x^2\frac{dy}{dx} + 4xy - 3y^2\frac{dy}{dx} = 0$. [1 marks]

Then $(2x^2 - 3y^2)\frac{dy}{dx} = -12x^3 - 4xy$ so $\frac{dy}{dx} = \frac{12x^3 + 4xy}{3y^2 - 2x^2}$. [2 marks]

(ii) The slope of the tangent line when $x = 1, y = -1$ is then $\frac{12-4}{3-2} = 8$ and the line has equation $y + 1 = 8(x - 1)$, so that $y = 8x - 9$. [4 marks]

(iii) If this line meets the curve $y = x^2 + 25$ at a point with the horizontal coordinate x, then $x^2 + 25 = 8x - 9$, giving the quadratic equation $x^2 - 8x + 16 = 0$. The only solution is $x = 4$ so the line and the curve meet in exactly one point. [4 marks]

13 (a) Put $u = 2 - \cos x$. Then $y = \ln(2 - \cos x) = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times (\sin x) = \frac{\sin x}{2 - \cos x}$$

[3 marks]

(b) $y = e^{2x+3}(3x - 1) = uv$ with $u = e^{2x+3}, v = 3x - 1$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{2x+3} \times 3 + (3x - 1) \times 2e^{2x+3} = e^{2x+3}(6x + 1).$$

[4 marks]

(c) In $u = \sin^3 x$, put $w = \sin x$. Then $u = w^3$. Hence $\frac{du}{dx} = \frac{du}{dw} \times \frac{dw}{dx} = 3w^2 \times \cos x = 3\sin^2 x \cos x$. Therefore $\frac{d}{dx}(\sin^3 x - 2x^5 + 4) = 3\sin^2 x\ \cos x - 10x^4$. [4 marks]

(d) Put $u = \sin^3 x$ and $w = \sin x$. Then $u = w^3$ and $\dfrac{du}{dx} = \dfrac{du}{dw} \times \dfrac{dw}{dx} = 3w^2 \times (\cos x) = 3\sin^2 x\ \cos x$. Setting also $v = 2x^2 - 3$, we have

$$\frac{d}{dx}\frac{\sin^3 x}{2x^2 - 3} = \frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2x^2 - 3) \times 3\sin^2 x \cos x - \sin^3 x \times 4x}{(2x^2 - 3)^2}$$

[4 marks]

14 (i) The stationary points occur where $f'(x) = 0$ and the inflection points where $f''(x) = 0$. Now $f'(x) = 3x^2 + 2x - 2$ and $f''(x) = 6x + 2$. [3 marks]

The inflection point occurs where $x = -\frac{1}{3} = 0.33$ and $f(x) = -\frac{20}{27} = -0.74$. [1 marks]

The stationary points are given by the quadratic formula as $x = \dfrac{-2 \pm \sqrt{4 + 24}}{6} = \dfrac{-1 \pm \sqrt{7}}{3} = 0.55$ or -1.22. These are respectively a local minimum, where $f''(x) > 0$, and a local maximum, where $f''(x) < 0$.

The corresponding values of $f(x)$ are -0.63 and 2.11. [2 marks]

(ii) The curve $y = x^3 + x^2 - 2x$ crosses the x-axis when $x^3 + x^2 - 2x = 0$. This happens when $x = 0$ or when $x^2 + x - 2 = 0$, giving $x = -2$ or $x = 1$ as well. [2 marks]

(iii) Using the information from (a) and (b), sketch the curve $y = x^3 + x^2 - 2x$.

[3 marks]

(iv) The total area bounded by the curve and the x-axis is made up of two pieces, one between $x = -2$ and $x = 0$, and the other between $x = 0$ and $x = 1$. These are found as $\left|\int_{-2}^{0} f(x)\,dx\right|$ and $\left|\int_{0}^{1} f(x)\,dx\right|$. Now $\int f(x)\,dx = \int (x^3 + x^2 - 2x)\,dx = \dfrac{x^4}{4} + \dfrac{1}{3}x^3 - x^2$, giving the first area as $|4 - 8/3 - 4| = 8/3$ and the second as $|1/4 + 1/3 - 1| = 5/12$ making a total of $37/12 = 3.08$. [4 marks]

15 (a) Substitute $u = x^4 + 3$. Then $du = 4x^3 dx$, so

$$I = \int x^3 \sin(x^4 + 3)\,dx = \int \tfrac{1}{4}\sin u\,du = -\tfrac{1}{4}\cos u + C = -\tfrac{1}{4}\cos(x^4 + 3) + C.$$

[4 marks]

(b) Substitute $t = \tan x$. Then $dt = \sec^2 x\,dx$, so

$$I = \int \tan^5 x \sec^2 x\ dx = \int t^5\,dt = \tfrac{t^6}{6} + C = \tfrac{1}{6}\tan^6 x + C.$$

[4 marks]

(c) Substitute $x = 2\sin t$. Then $dx = 2\cos t\,dt$. Therefore

$$I = \int_0^1 \frac{dx}{\sqrt{4 - x^2}} = \int_{x=0}^{x=1} \frac{2\cos t}{\sqrt{4 - 4\sin^2 t}}\,dt = \int_{x=0}^{x=1} \frac{2\cos t}{\sqrt{4\cos^2 t}}\,dt = \int_{x=0}^{x=1} dt = [t]_{x=0}^{x=1}$$

Now when $x = 0$ we have $t = 0$ and when $x = 1$ we have $1 = 2\sin t$ so that $\sin t = \frac{1}{2}$ and $t = \frac{\pi}{6}$. Consequently $I = [t]_{t=0}^{t=\pi/6} = (\dfrac{\pi}{6} - 0) = 0.52$. [7 marks]