MATH011 **Exam Solutions** **Sept 2006**

1. (a) $\dfrac{a^9b^{-3}c^8}{a^7(bc^2)^4} = \dfrac{a^9b^{-3}c^8}{a^7b^4c^8} = \dfrac{a^2}{b^7}$ **[2]**

(b) $\dfrac{16x^2-1}{16x^2+8x+1} = \dfrac{(4x-1)(4x+1)}{(4x+1)^2} = \dfrac{4x-1}{4x+1}$. **[2]**

2. $\dfrac{1}{y-7} - \dfrac{7}{y^2-7y} = \dfrac{y-7}{y(y-7)} = \dfrac{1}{y}$ **[4]**

3. (a) $x^2 - x - 72 = (x-9)(x+8)$ so solutions are $x = 9, x = -8$. **[2]**

(b) Using the quadratic formula $x = \dfrac{2 \pm \sqrt{4 - 4\times 8 \times (-21)}}{16} = \dfrac{2 \pm 26}{16} = \dfrac{7}{4}$ or $-\dfrac{3}{2}$ **[2]**

4. (a) $y = 3x - 6$ represents a straight line with slope 3 meeting the y-axis at $y = -6$. **[2]**
(b) $y = x^2 + 6x + 8$ is a quadratic curve, which is U-shaped, crossing the y-axis at $y = 8$ and the x-axis at $x = -2, x = -4$. The curve is symmetric about the line $x = -6/2 = -3$. The vertex is at $x = -3, y = -1$. **[3]**
(c) $y = |x^2 + 6x + 8|$ is given from (b) by reflecting the part below the x-axis in the x-axis. **[2]**

5. Put $y = \dfrac{2x+7}{1-3x}$, and solve for x in terms of $y = f(x)$. Then $y(1-3x) = 2x+7$ so $y - 3xy = 2x + 7$, giving $x(3y+2) = y - 7$.
Thus $x = f^{-1}(y) = \dfrac{y-7}{3y+2}$ and so $f^{-1}(x) = \dfrac{x-7}{3x+2}$. **[3]**

6. (a) Either use the formula $a\dfrac{1-r^n}{1-r}$ with $r = -5, a = -5, n = 5$, giving $-5 \times \dfrac{(-5)^5 - 1}{-5-1} = -2605$ or simply add up the 5 terms. **[3]**

(b) The formula is $\dfrac{a}{1-r}$. **[1]**

Here $a = \frac{7}{10}, r = \frac{7}{10}$, giving the sum as $\dfrac{\frac{7}{10}}{1-\frac{7}{10}} = \dfrac{\frac{7}{10}}{\frac{3}{10}} = \dfrac{7}{3}$. **[2]**

7. (a) $\displaystyle\lim_{n\to\infty} \frac{2n^2 - 7n}{5n^2 + 6n - 1} = \lim_{n\to\infty} \frac{2 - \frac{7}{n}}{5 + \frac{6}{n} - \frac{1}{n^2}} \to \frac{2}{5}$ as $n \to \infty$. **[2]**

(b) Putting $x = 4$ in the bottom of the fraction gives 0, as also in the top. Factorise to write

$$\frac{x^2 - x - 12}{x^2 - 16} = \frac{(x-4)(x+3)}{(x-4)(x+4)} = \frac{x+3}{x+4}$$

Now put $x = 4$ to get the limit $\frac{7}{8}$. **[2]**

8. **(a)** Put $u = 3x - 8$. Then $y = (3x-8)^7 = u^7$ so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 7u^6 \times 3 = 21(3x-8)^6.$$ **[2]**

(b) Put $u = x^4 + 3$. Then $y = (x^4+3)^{\frac{5}{4}} = u^{\frac{5}{4}}$ and

$$\frac{dy}{dx} = \frac{5}{4}u^{\frac{1}{4}} \times \frac{du}{dx} = \frac{5}{4}u^{\frac{1}{4}} \times 4x^3 = 5x^3(x^4+3)^{\frac{1}{4}}.$$ **[3]**

(c) $y = x^8 \cos x = uv$ with $u = x^8, v = \cos x$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^8(-\sin x) + 8x^7 \cos x = -x^8 \sin x + 8x^7 \cos x.$$ **[3]**

9. The slope of the tangent is the value of $\frac{dy}{dx}$ at $x = -1$. Now $\frac{dy}{dx} = 12x^3$, so the slope is -12. When $x = -1$ we have $y = 5$, so the tangent line has equation $y - 5 = -12(x+1)$, giving $y = -12x - 7$. **[3]**

10. **(a)** $\int (6 - 3x^5 - \sin x)\, dx = 6\int 1\, dx - 3\int x^5\, dx - \int \sin x\, dx = 6x - \frac{1}{2}x^6 + \cos x + C$ **[4]**

(b) $\int e^{-5x}\, dx = \frac{1}{-5}e^{-5x} + C = -\frac{1}{5}e^{-5x} + C.$ **[2]**

11. **(a)** $\int_0^{\pi/14} \cos 7x\, dx = \left[\frac{1}{7}\sin 7x\right]_0^{\pi/14} = \frac{1}{7}\sin(\pi/2) - \frac{1}{7}\sin 0 = \frac{1}{7}.$ **[3]**

(b) Substitute $u = 5x - 14$ so that $du = 5dx$. Then

$$I = \int_3^6 \frac{5}{5x-14}\, dx = \int_{x=3}^{x=6} \frac{1}{u}\, du = \int_{u=1}^{u=16} \frac{1}{u}\, du = [\ln u]_1^{16} = \ln 16 - \ln 1 = \ln 16$$ **[3]**

12. **(i)** Differentiate the LHS to get

$$3 \times 3x^2 + (x \times 2y\frac{dy}{dx} + y^2) - 2 \times 4y^3\frac{dy}{dx}$$ **[4]**

The RHS has derivative 0 giving the equation $9x^2 + 2xy\dfrac{dy}{dx} + y^2 - 8y^3\dfrac{dy}{dx} = 0$. **[1]**

Then $(2xy - 8y^3)\dfrac{dy}{dx} = -9x^2 - y^2$ so $\dfrac{dy}{dx} = \dfrac{9x^2 + y^2}{8y^3 - 2xy}$. **[2]**

(ii) The slope of the tangent line when $x = -1, y = 1$ is then $\dfrac{9+1}{8+2} = 1$ and the line has equation $y - 1 = x + 1$, so that $y = x + 2$. **[4]**

(iii) If this line meets the curve $y = x^2 + 5x + 6$ at a point with the horizontal coordinate x, then $x^2 + 5x + 6 = x + 2$, giving the quadratic equation $x^2 + 4x + 4 = 0$. The only solution is $x = -2$ so the line and the curve meet in exactly one point. **[4]**

13. (a) Put $u = 5 - \sin x$. Then $y = \ln(5 - \sin x) = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times (-\cos x) = -\frac{\cos x}{5 - \sin x}$$ **[3]**

(b) $y = e^{5x+8}(7 - 3x) = uv$ with $u = e^{5x+8}, v = 7 - 3x$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{5x+8} \times (-3) + (7 - 3x) \times 5e^{5x+8} = e^{5x+8}(32 - 15x).$$

[4]

(c) In $u = \cos^4 x$, put $w = \cos x$. Then $u = w^4$. Hence $\dfrac{du}{dx} = \dfrac{du}{dw} \times \dfrac{dw}{dx} = 4w^3 \times (-\sin x) = -4\cos^3 x \sin x$. Therefore $\dfrac{d}{dx}(5x^7 - 10 - \cos^4 x) = 35x^6 + 4\cos^3 x \sin x$. **[4]**

(d) Put $v = \sin^3 x$ and $w = \sin x$. Then $v = w^3$ and $\dfrac{dv}{dx} = \dfrac{dv}{dw} \times \dfrac{dw}{dx} = 3w^2 \times \cos x = 3\sin^2 x\ \cos x$. Setting also $u = 2x^4 - 7$, we have

$$\frac{d}{dx}\frac{2x^4 - 7}{\sin^3 x} = \frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\sin^3 x \times 8x^3 - (2x^4 - 7) \times 3\sin^2 x\ \cos x}{\sin^6 x} = \frac{8x^3 \sin x - 3(2x^4 - 7)\cos x}{\sin^4 x}$$ **[4]**

14. (i) The stationary points occur where $f'(x) = 0$ and the inflection points where $f''(x) = 0$. Now $f'(x) = 3x^2 + 2x - 12$ and $f''(x) = 6x + 2$. **[3]**

The inflection point occurs where $x = -\frac{1}{3} = -0.33$ and $f(x) = 4\frac{2}{27} = 4.07$. **[1]**

The stationary points are given by the quadratic formula as $x = \dfrac{-2 \pm \sqrt{4 + 144}}{6} = \dfrac{-1 \pm \sqrt{37}}{3} = -2.36$ or 1.69. These are respectively a local maximum, where $f''(x) < 0$, and a local minimum, where $f''(x) > 0$.

The corresponding values of $f(x)$ are 20.75 and -12.58. **[2]**

(ii) The curve $y = x^3 + x^2 - 12x$ crosses the x-axis when $x^3 + x^2 - 12x = 0$. This happens when $x = 0$ or when $x^2 + x - 12 = 0$, giving $x = -4$ or $x = 3$ as well. **[2]**

(iii) Using the information from (a) and (b), sketch the curve $y = x^3 + x^2 - 12x$. **[3]**

(iv) The total area bounded by the curve and the x-axis is made up of two pieces, one between $x = -4$ and $x = 0$, and the other between $x = 0$ and $x = 3$. These are found as $\left|\int_{-4}^{0} f(x)\,dx\right|$ and $\left|\int_{0}^{3} f(x)\,dx\right|$. Now $\int f(x)\,dx = \int (x^3 + x^2 - 12x)\,dx = \dfrac{x^4}{4} + \dfrac{1}{3}x^3 - 6x^2$, giving the first area as $|-64 + 64/3 + 96| = 160/3$ and the second as $|81/4 + 9 - 54| = 99/4$ making a total of $937/12 = 78.08$. **[4]**

15. (a) Substitute $u = x^6 + 4$. Then $du = 6x^5 dx$, so

$$I = \int x^5 \cos(x^6 + 4)\,dx = \int \frac{1}{6} \cos u\,du = \frac{1}{6} \sin u + C = \frac{1}{6} \sin(x^6 + 4) + C. \qquad [4]$$

(b) Substitute $t = \tan x$. Then $dt = \sec^2 x\,dx$, so

$$I = \int \tan^8 x \sec^2 x\ dx = \int t^8\,dt = \frac{t^9}{9} + C = \frac{1}{9} \tan^9 x + C. \qquad [4]$$

(c) Substitute $x = \frac{2}{5} \sin t$. Then $dx = \frac{2}{5} \cos t\,dt$. Therefore

$$I = \int_0^{\sqrt{2}/5} \frac{dx}{\sqrt{4 - 25x^2}} = \int_{x=0}^{x=\sqrt{2}/5} \frac{\frac{2}{5}\cos t}{\sqrt{4 - 4\sin^2 t}}\,dt = \int_{x=0}^{x=\sqrt{2}/5} \frac{\frac{2}{5}\cos t}{\sqrt{4\cos^2 t}}\,dt = \frac{1}{5} \int_{x=0}^{x=\sqrt{2}/5} dt = [t]_{x=0}^{x=\sqrt{2}/5}$$

Now when $x = 0$ we have $t = 0$ and when $x = \sqrt{2}/5$ we have $\frac{\sqrt{2}}{5} = \frac{2}{5} \sin t$ so that $\sin t = \frac{\sqrt{2}}{2}$ and $t = \frac{\pi}{4}$. Consequently $I = \dfrac{1}{5}[t]_{t=0}^{t=\pi/4} = \dfrac{1}{5}(\dfrac{\pi}{4} - 0) = \dfrac{\pi}{20} = 0.16$. **[7]**