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SECTION A

**1.** Simplify:

(a) $\dfrac{(a^3b)^5c^{-4}}{a^{15}b^4c^3}$ (b) $\dfrac{1-16x^2}{1+8x+16x^2}$

[4 marks]

**2.** Write $\dfrac{2}{2x+5}+\dfrac{5}{2x^2+5x}$ as a single fraction, and simplify it as far as possible.

[4 marks]

**3.** Solve the following quadratic equations:

(a) $x^2+3x-28=0$ (b) $12x^2+13x-35=0$

[4 marks]

**4.** Sketch the graph of each of the functions:

(a) $y=-2x+4$ (b) $y=x^2-4x-12$ (c) $y=|x^2-4x-12|$

[7 marks]

**5.** Given that $f(x)=\dfrac{3-2x}{4x+5}$, obtain an expression for the inverse function $f^{-1}(x)$.

[3 marks]

**6.** (a) Find the sum of the geometric series $\sum_{n=1}^{4}(-5)^n$ .

(b) Write down the formula for the sum of the infinite geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ with first term $a$ and common ratio $r$, when $|r|<1$.

Hence show that $\sum_{n=1}^{\infty}\left(\frac{2}{11}\right)^n=\frac{2}{9}$ .

[6 marks]

**7.** Evaluate the following limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{4n^2-7}{9-3n-2n^2}$ (b) $\displaystyle\lim_{x\to 6} \frac{x^2-36}{x^2-2x-24}$

[4 marks]

**8.** Differentiate with respect to $x$:

(a) $(5x-6)^4$ (b) $(x^7+2)^{6/7}$ (c) $x^3\cos x$

[8 marks]

**9.** Write down the equation of the tangent line to the curve $y=-x^3+4$ at the point where $x=2$.

[3 marks]

**10.** Find the indefinite integrals:

(a) $\displaystyle\int (\sin x + 3x^5 - 4)\,dx$

(b) $\displaystyle\int e^{-4x}\,dx$

[6 marks]

**11.** Evaluate the definite integrals:

(a) $\displaystyle\int_0^{\pi/10} \cos 5x\,dx$

(b) $\displaystyle\int_0^4 \frac{6}{6x+1}\,dx$ [Substitute $u=6x+1$]

[6 marks]

SECTION B

**12.** The function $y = y(x)$ satisfies the equation

$$2x^3 + 3xy^2 - 7y^3 = 15\,.$$

(i) By differentiating both sides of this equation with respect to $x$ find an equation relating $x, y$ and $\dfrac{dy}{dx}$.

Hence show that

$$\frac{dy}{dx} = \frac{2x^2 + y^2}{7y^2 - 2xy}\,.$$

(ii) Use this to show that the tangent line to the curve

$$2x^3 + 3xy^2 - 7y^3 = 15$$

at the point where $x = 2$ and $y = 1$ has equation

$$y = 3x - 5\,.$$

(iii) Show that this line meets the curve with equation

$$y = x^2 + x - 4$$

in exactly one point.

[15 marks]

**13.** Differentiate the functions:

(a) $\ln(3 - \sin x)$ (b) $e^{4x+5}(2 - 3x)$

(c) $\cos^6 x - 5x^7 + 8$ (d) $\dfrac{4 - 2x^5}{\sin^3 x}$

[15 marks]

**14.** (i) Find the stationary points and the inflection point of the function

$$f(x) = x^3 + 2x^2 - 15x\,,$$

in each case giving the values of $x$ and $f(x)$ to 2 decimal places.

Determine also the nature of the stationary points.

(ii) Find the three points at which the curve $y = x^3 + 2x^2 - 15x$ crosses the $x$-axis.

(iii) Using the information from (a) and (b), sketch the curve

$$y = x^3 + 2x^2 - 15x\,.$$

(iv) Calculate the total area bounded by the curve and the $x$-axis.

[15 marks]

**15.** Find the indefinite integrals:

(a) $\displaystyle\int x^5 \cos(x^6 + 8)\,dx$ [Substitute $u = x^6 + 8$]

(b) $\displaystyle\int \tan^9 x\ \sec^2 x\ dx$ [Substitute $t = \tan x$]

Evaluate the definite integral:

(c) $\displaystyle\int_0^{\frac{2\sqrt{2}}{5}} \frac{dx}{\sqrt{16 - 25x^2}}$ [Substitute $x = \dfrac{4}{5}\sin t$]

[15 marks]

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## Formulae Handbook

This handbook is designed for examination purposes, to be used in the Semester Examination. The information contained below is not exhaustive of all the formulae given in the lectures which you may need.

**1.** Solutions to the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

**2. Power Laws**

(a) $b^0 = 1$ (b) $b^1 = b$ (c) $b^{-n} = \dfrac{1}{b^n}$ (d) $b^{\frac{1}{n}} = \sqrt[n]{b}$

(e) $b^m b^n = b^{m+n}$ (f) $\dfrac{b^m}{b^n} = b^m b^{-n} = b^{m-n}$

(g) $(b^m)^n = b^{mn} = (b^n)^m$

(h) $(ab)^n = a^n b^n$ (i) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} = a^n b^{-n}$

**3. Logarithms**

(a) $M = b^x \iff x = \log_b M$

(b) $\log_b(xy) = \log_b x + \log_b y$

(c) $\log_b \dfrac{x}{y} = \log_b x - \log_b y$

(d) $\log_b(x^n) = n \log_b x$

**4. Series**

(a) The $n$th term of an arithmetic series with first term $a$ and common difference $d$ is $a + (n-1)d$. The sum of its first $n$ terms is $na + \frac{1}{2}n(n-1)d$.

(b) The $n$th term of a geometric series with first term $a$ and common ratio $r$ is $ar^{n-1}$. The sum of its first $n$ terms is $a \times \dfrac{r^n - 1}{r - 1}$ .

The sum to infinity, when $|r| < 1$, is $\dfrac{a}{1-r}$.

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## 5. Trigonometric functions

$$\begin{aligned}
\tan x &= \tfrac{\sin x}{\cos x} \\
\cot x &= \tfrac{\cos x}{\sin x} \\
\sec x &= \tfrac{1}{\cos x} \\
\operatorname{cosec} x &= \tfrac{1}{\sin x}
\end{aligned}$$

## 6. Derivatives

(a) If $y = x^n$ where $n$ is constant then $\frac{dy}{dx} = nx^{n-1}$.

(b) If $y = uv$, where $u$ and $v$ are functions of $x$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

(c) If $y = \frac{u}{v}$, where $u$ and $v$ are functions of $x$, then $\frac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

(d) If $y$ is a function of $u$ and $u$ is a function of $x$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

(e) $\frac{d}{dx}(y^r) = \frac{d}{dy}(y^r) \times \frac{dy}{dx} = ry^{r-1}\frac{dy}{dx}$.

| Function | Derivative |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\ln x$ | $\frac{1}{x}$ |
| $e^x$ | $e^x$ |

## 7. Integrals

$$\begin{aligned}
\int x^n \, dx &= \tfrac{1}{n+1}x^{n+1} \\
\int \tfrac{1}{x} \, dx &= \ln x \\
\int e^x \, dx &= e^x \\
\int \cos x \, dx &= \sin x \\
\int \sin x \, dx &= -\cos x \\
\int \sec^2 x \, dx &= \tan x
\end{aligned}$$