MATH011 Exam Solutions Jan 2006

1. (a) $\dfrac{x^7(y^2z)^4}{x^{-3}y^8z^5} = \dfrac{x^7y^8z^4}{x^{-3}y^8z^5} = x^{10}z^{-1}$ **[2]**

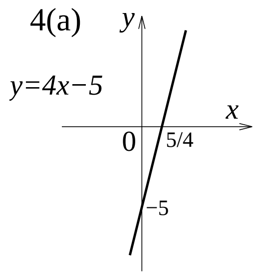
(b) $\dfrac{4b^2-1}{4b^2-4b+1} = \dfrac{(2b-1)(2b+1)}{(2b-1)^2} = \dfrac{2b+1}{2b-1}.$ **[2]**

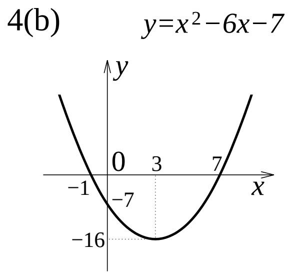
2. $\dfrac{1}{a-3} - \dfrac{3}{a^2-3a} = \dfrac{a-3}{a(a-3)} = \dfrac{1}{a}$ **[4]**

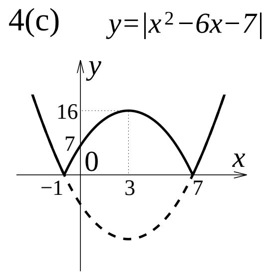
3. (a) $x^2 - x - 42 = (x-7)(x+6)$ so solutions are $x = 7, x = -6$. **[2]**

(b) Using the quadratic formula $x = \dfrac{13 \pm \sqrt{169 - 4 \times 10 \times (-3)}}{20} = \dfrac{13 \pm 17}{20} = \dfrac{3}{2}$ or $-\dfrac{1}{5}$ **[2]**

4. (a) $y = 4x - 5$ represents a straight line with slope 4 meeting the y-axis at $y = -5$. **[2]**
(b) $y = x^2 - 6x - 7$ is a quadratic curve, which is U-shaped, crossing the y-axis at $y = -7$ and the x-axis at $x = -1, x = 7$. The curve is symmetric about the line $x = -(-6)/2 = 3$. The vertex is at $x = 3, y = -16$. **[3]**
(c) $y = |x^2 - 6x - 7|$ is given from (b) by reflecting the part below the x-axis in the x-axis. **[2]**







5. Put $y = \dfrac{1-3x}{2x+7}$, and solve for x in terms of $y = f(x)$. Then $(2x+7)y = 1 - 3x$ so $2xy + 7y = 1 - 3x$, giving $x(2y+3) = 1 - 7y$.
Thus $x = f^{-1}(y) = \dfrac{1-7y}{2y+3}$ and so $f^{-1}(x) = \dfrac{1-7x}{2x+3}$. **[3]**

6. (a) Either use the formula $a\dfrac{1-r^n}{1-r}$ with $r = -4, a = -4, n = 5$, giving $-4 \times \dfrac{(-4)^5 - 1}{-4-1} = -820$ or simply add up the 5 terms. **[3]**

(b) The formula is $\dfrac{a}{1-r}$. **[1]**

Here $a = \frac{5}{9}, r = \frac{5}{4}$, giving the sum as $\dfrac{\frac{5}{9}}{1-\frac{5}{9}} = \dfrac{\frac{5}{9}}{\frac{4}{9}} = \dfrac{5}{4}$. **[2]**

7. (a) $\displaystyle\lim_{n\to\infty} \frac{4+5n-3n^2}{2n^2+7} = \lim_{n\to\infty} \frac{\frac{4}{n^2}+\frac{5}{n}-3}{2+\frac{7}{n^2}} \to \frac{-3}{2} = -\frac{3}{2}$ as $n \to \infty$. **[2]**

(b) Putting $x = 3$ in the bottom of the fraction gives 0, as also in the top. Factorise to write

$$\frac{x^2+x-12}{x^2-9} = \frac{(x-3)(x+4)}{(x-3)(x+3)} = \frac{x+4}{x+3}$$

Now put $x = 3$ to get the limit $\frac{7}{6}$. **[2]**

8. (a) Put $u = 4x+3$. Then $y = (4x+3)^6 = u^6$ so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 6u^5 \times 4 = 24(4x+3)^5.$$ **[2]**

(b) Put $u = x^3 - 2$. Then $y = (x^3-2)^{\frac{5}{3}} = u^{\frac{5}{3}}$ and

$$\frac{dy}{dx} = \frac{5}{3}u^{\frac{2}{3}} \times \frac{du}{dx} = \frac{5}{3}u^{\frac{2}{3}} \times 3x^2 = 5x^2(x^3-2)^{\frac{2}{3}}.$$ **[3]**

(c) $y = x^7 \sin x = uv$ with $u = x^7, v = \sin x$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x^7 \cos x + 7x^6 \sin x.$$ **[3]**

9. The slope of the tangent is the value of $\dfrac{dy}{dx}$ at $x = 1$. Now $\dfrac{dy}{dx} = -6x^2$, so the slope is -6. When $x = 1$ we have $y = 3$, so the tangent line has equation $y - 3 = -6(x-1)$, giving $y = -6x + 9$. **[3]**

10. (a) $\displaystyle\int(\cos x - 2x^3 + 5)\,dx = \int \cos x\,dx - 2\int x^3\,dx + 5\int 1\,dx = \sin x - \frac{1}{2}x^4 + 5x + C$ **[4]**

(b) $\displaystyle\int e^{7x}\,dx = \frac{1}{7}e^{7x} + C.$ **[2]**

11. (a) $\displaystyle\int_0^{\frac{\pi}{8}} \sin 8x\,dx = \left[-\frac{1}{8}\cos 8x\right]_0^{\frac{\pi}{8}} = -\frac{1}{8}\cos\pi + \frac{1}{8}\cos 0 = \frac{1}{4}.$ **[3]**

(b) Substitute $u = 3x - 5$ so that $du = 3dx$. Then

$$I = \int_2^7 \frac{3}{3x-5}\,dx = \int_{x=2}^{x=6} \frac{1}{u}\,du = \int_{u=1}^{u=16} \frac{1}{u}\,du = [\ln u]_1^{16} = \ln 16 - \ln 1 = \ln 16$$ **[3]**

12. **(i)** Differentiate the LHS to get

$$2 \times 4x^3 - (x^2 \frac{dy}{dx} + 2xy) - 3 \times 3y^2 \frac{dy}{dx} \quad \textbf{[4]}$$

The RHS has derivative 0 giving the equation $8x^3 - x^2 \frac{dy}{dx} - 2xy - 9y^2 \frac{dy}{dx} = 0$. **[1]**

Then $(x^2 + 9y^2)\frac{dy}{dx} = 8x^3 - 2xy$ so $\frac{dy}{dx} = \frac{8x^3 - 2xy}{x^2 + 9y^2}$. **[2]**

(ii) The slope of the tangent line when $x = 1, y = -1$ is then $\frac{8+2}{1+9} = 1$ and the line has equation $y + 1 = x - 1$, so that $y = x - 2$. **[4]**

(iii) If this line meets the curve $y = x^2 - 3x + 2$ at a point with the horizontal coordinate x, then $x^2 - 3x + 2 = x - 2$, giving the quadratic equation $x^2 - 4x + 4 = 0$. The only solution is $x = 2$ so the line and the curve meet in exactly one point. **[4]**

13. **(a)** Put $u = \cos x + 2$. Then $y = \ln(\cos x + 2) = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times (-\sin x) = -\frac{\sin x}{\cos x + 2} \quad \textbf{[3]}$$

(b) $y = e^{3-2x}(5x + 4) = uv$ with $u = e^{3-2x}, v = 5x + 4$.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = e^{3-2x} \times 5 + (5x + 4) \times (-2)e^{3-2x} = -e^{3-2x}(10x + 3).$$

[4]

(c) In $u = \sin^5 x$, put $w = \sin x$. Then $u = w^5$. Hence $\frac{du}{dx} = \frac{du}{dw} \times \frac{dw}{dx} = 5w^4 \times \cos x = 5\sin^4 x \cos x$. Therefore $\frac{d}{dx}(\sin^5 x + 2x^4 - 7) = 5\sin^4 x\ \cos x + 8x^3$. **[4]**

(d) Put $v = \cos^4 x$ and $w = \cos x$. Then $v = w^4$ and $\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx} = 4w^3 \times (-\sin x) = -4\cos^3 x\ \sin x$. Setting also $u = 4x^3 - 5$, we have

$$\frac{d}{dx}\frac{4x^3 - 5}{\cos^4 x} = \frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

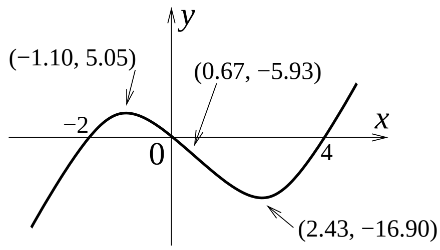
$$= \frac{\cos^4 x \times 12x^2 - (4x^3 - 5)(-4\cos^3 x\ \sin x)}{\cos^8 x} = \frac{12x^2 \cos x + 4(4x^3 - 5)\sin x}{\cos^5 x} \quad \textbf{[4]}$$

14. **(i)** The stationary points occur where $f'(x) = 0$ and the inflection points where $f''(x) = 0$. Now $f'(x) = 3x^2 - 4x - 8$ and $f''(x) = 6x - 4$. **[3]**

The inflection point occurs where $x = \frac{2}{3} = 0.67$ and $f(x) = -\frac{160}{27} = -5.93$. **[1]**

The stationary points are given by the quadratic formula as $x = \dfrac{4 \pm \sqrt{16+96}}{6} = \dfrac{2 \pm 2\sqrt{7}}{3} = 2.43$ or -1.10. These are respectively a local minimum, where $f''(x) > 0$, and a local maximum, where $f''(x) < 0$.
The corresponding values of $f(x)$ are -16.90 and 5.05. **[2]**
(ii) The curve $y = x^3 - 2x^2 - 8x$ crosses the x-axis when $x^3 - 2x^2 - 8x = 0$. This happens when $x = 0$ or when $x^2 - 2x - 8 = 0$, giving $x = -2$ or $x = 4$ as well. **[2]**
(iii) Using the information from (a) and (b), sketch the curve $y = x^3 - 2x^2 - 8x$. **[3]**
(iv) The total area bounded by the curve and the x-axis is made up of two pieces, one between $x = -2$ and $x = 0$, and the other between $x = 0$ and $x = 4$. These are found as $\left|\int_{-2}^{0} f(x)\,dx\right|$ and $\left|\int_{0}^{4} f(x)\,dx\right|$. Now $\int f(x)\,dx = \int (x^3 - 2x^2 - 8x)\,dx = \dfrac{x^4}{4} - \dfrac{2}{3}x^3 - 4x^2$, giving the first area as $|4 + 16/3 - 16| = 20/3$ and the second as $|64 - 128/3 - 64| = 128/3$ making a total of $148/3 = 49.33$. **[4]**

14(iii)



15. **(a)** Substitute $u = x^5 - 6$. Then $du = 5x^4 dx$, so

$$I = \int x^4 \sin(x^5 - 6)\,dx = \int \frac{1}{5}\sin u\,du = -\frac{1}{5}\cos u + C = -\frac{1}{5}\cos(x^5 - 6) + C. \qquad [4]$$

(b) Substitute $t = \tan x$. Then $dt = \sec^2 x\,dx$, so

$$I = \int \tan^6 x \sec^2 x\ dx = \int t^6\,dt = \frac{t^7}{7} + C = \frac{1}{7}\tan^7 x + C. \qquad [4]$$

(c) Substitute $x = \frac{2}{3}\sin t$. Then $dx = \frac{2}{3}\cos t\,dt$. Therefore

$$I = \int_0^{1/3} \frac{dx}{\sqrt{4 - 9x^2}} = \int_{x=0}^{x=1/3} \frac{\frac{2}{3}\cos t}{\sqrt{4 - 4\sin^2 t}}\,dt = \int_{x=0}^{x=1/3} \frac{\frac{2}{3}\cos t}{\sqrt{4\cos^2 t}}\,dt = \frac{1}{3}\int_{x=0}^{x=1/3} dt = [t]_{x=0}^{x=1/3}$$

Now when $x = 0$ we have $t = 0$ and when $x = 1/3$ we have $1/3 = \frac{2}{3}\sin t$ so that $\sin t = \frac{1}{2}$ and $t = \frac{\pi}{6}$. Consequently $I = \dfrac{1}{3}[t]_{t=0}^{t=\pi/6} = \dfrac{1}{3}(\dfrac{\pi}{6} - 0) = \dfrac{\pi}{18} = 0.17$. **[7]**