1. (a) A sports league contains 16 teams. Each team plays each other team, at home and away. Determine the number of games played by each team. The first three teams at the end of the season are awarded a gold, silver or bronze cup repectively. Calculate the total number of possible cup-winners. Three teams are relegated at the end of the season. Calculate the number of ways in which this can occur. Determine the number of ways in which the 16 teams can be formed into 4 groups of 4 teams each.

(b) Let T be an equilateral triangle with each side of length 2 cm. Show that if five points are placed on or inside T, at least two of the points are within 1 cm of each other.

(c) Let X be the set $\{1, 2, 3, \ldots, 100\}$. Determine the minimum number, k, of integers selected from X which will ensure that two of the selected k numbers add up to 120. Give an example of a set with $k - 1$ integers no two of which add up to 120.

2. Obtain a formula for the number of ways of distributing r identical objects into n distinct containers.

Determine the number of integer solutions of $x + y + z + u + v = 34$ when

(a) each of x, y, z, u, v are non-negative;

(b) each of x, y, z, u, v are positive;

(c) each of x, y, z are ≥ 4 and u, v are ≥ 7;

(d) each of x, y, z, u, v are ≥ -2;

(e) each of x, y, z, u, v are non-negative and are less than 5;

(f) each of x, y, z, u, v are non-negative and are less than 8;

(g) $0 \leq x \leq 13$, $0 \leq y \leq 13$, $0 \leq z \leq 14$, $3 \leq u \leq 16$, and $6 \leq v \leq 21$.

3. State Philip Hall's Assignment Theorem.

Construct a family of sets which has a system of distinct representatives if and only if the pruned chessboard shown below has a perfect cover. Decide whether the chessboard has a perfect cover or not.

□	□	■	■	■	■
■	□	□	■	□	□
□	□	□	□	□	□
■	□	□	■	□	■
■	■	□	□	□	■

(Here a ■ denotes a square that has been deleted).

There are 8 vacancies and 8 applicants. The set of qualified applicants for each of the posts is respectively $\{a_1, a_5\}$, $\{a_2, a_3, a_5\}$, $\{a_3, a_8\}$, $\{a_4, a_7\}$, $\{a_3, a_5, a_6\}$, $\{a_3, a_5\}$, $\{a_6, a_8\}$, and $\{a_2, a_5, a_8\}$. Determine the largest number of vacancies which can be filled by qualified applicants.

4. Define a *rook polynomial* and calculate the rook polynomial for a 3×3 chessboard. Give rules which will enable the rook polynomial of any board to be calculated.

The Head of Mathematics has five lecturers available, a, b, c, d and e, to teach Algebra (denoted A), Basic Analysis (B), Combinatorics (C), Discrete Mathematics (D) and Electomagnetism (E). Lecturer a refuses to teach B, lecturer b refuses to teach A and C, c refuses to teach A and B, d refuses to teach C and E and e refuses to teach C. Calculate the number of ways in which the Head of Department can complete his lecture schedule.

5. (a) Let D_n be determinant of the $n \times n$ matrix with 2's down the main diagonal, 1's in the diagonals above and below the main diagonal, and 0's elesewhere, so that, for example,

$$D_5 = det \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Find a linear recursion formula for D_n, and hence find a general formula for the values of D_n.

(b) Using generating functions, or otherwise, find a expression for the n-th term of the sequence $\{a_n\}$ determined by $a_0 = 1 = a_1 = a_2$ and

$$a_{n+3} = 4a_{n+2} + 4a_{n+1} - 16a_n \text{ for } n \geq 0.$$

6. For $n \geq 0$ evenly distribute $2n$ points on the circumference of a circle, and label the points clockwise with the integers $1, 2, \ldots, 2n$ in order. Let a_n be the number of ways in which these points can be paired off by n chords of the circle with no two chords intersecting and no two chords joining the same vertex. Sketch the possibilities for $n = 3$. Show that there is a chord joining the first vertex to a vertex whose number is an even integer. Find a recurrence relation for a_n, and use generating functions to find an explicit formula for a_n.

7. Establish the generating function $p(x)$ which enumerates the number of partitions of the positive integer n. Find also the function $p_o(n)$ which enumerates the number of partitions of n into odd parts, and the function $p_d(x)$ to enumerate the partitions of n with distinct parts. For any positive integer n, show that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Show that the number of partitions of a positive integer n into parts none of which occurs more than twice is equal to the number of partitions of n where no part is divisible by 3.

Now let $p_1(n,k)$ denote the number of partitions of $2n+k$ with one of the parts equal to $n+k$, and let $p_2(n,k)$ denote the number of partitions of $2n+k$ with precisely $n+k$ parts. Use Ferrer's graphs to show that $p_1(n,k) = p_2(n,k)$ and that $p_1(n,k) = p(n)$. Deduce thet the number of partitions of $2n+k$ into precisely $n+k$ parts is independent of k.

8. Define the term *symmetric function.* For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities. Define the elementary symmetric function σ_λ and the power sum symmetric function π_λ for any partition λ. Express each of the following functions in terms of (a) the elementary symmetric functions and (b) the power sum functions:

(i) $f(x,y,z) = x^3 + y^3 + z^3$;

(ii) $g(x,y,z) = (x-y)^2 + (y-z)^2 + (z-x)^2$; and

(iii) $xyzh(x,y,z)$ where

$$h(x,y,z) = \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{x}{z} + \frac{z}{x}.$$