Throughout this paper, f^n denotes the n'th iterate of f and $\widetilde{f}^n$ denotes the n'th iterate of $\widetilde{f}$.

1. a) Define a *lift* $\widetilde{f} : \mathbf{R} \to \mathbf{R}$ of a continuous map $f : S^1 \to S^1$. Let $\widetilde{f}$ and $\widetilde{g}$ be lifts of continuous maps f and $g : S^1 \to S^1$. Find lifts of

(i) $f.g$, (ii) $f \circ g$.

b) Define $\deg(f)$ in terms of $\widetilde{f}$. Show that if $\deg(f) = d$ then

$$\widetilde{f}(\theta + n) - \widetilde{f}(\theta) = dn \text{ for all } \theta \in \mathbf{R}.$$

By considering suitable lifts or otherwise, find $\deg(f.g)$ and $\deg(f \circ g)$ in terms of $\deg(f)$ and $\deg(g)$.

c) Give the relationship between $\deg(f_0)$ and $\deg(f_1)$ if f_0 and f_1 are homotopic. Using this or otherwise, find $\deg(f_1)$ where

(i) $f_1(z) = \dfrac{(z - \frac{1}{2})^3}{z^2(1 - \frac{1}{2}z)^3}$, (ii) $f_1(z) = \dfrac{(z - \frac{1}{2})(2 - z)}{(1 - \frac{1}{2}z)(1 - 2z)}$.

[20 marks]

2. a) Let $f : S^1 \to S^1$ be continuous with lift $\widetilde{f} : \mathbf{R} \to \mathbf{R}$. Give the property of $\widetilde{f}$ which is equivalent to f being a *covering*. Give the possible values of the degree of f when f is a covering. If f is complex differentiable, give a condition on the derivative f' which implies that f is a covering. Determine which of the following are coverings.

(i) $f(z) = \dfrac{z - \frac{1}{3}}{1 - \frac{1}{3}z}$, (ii) $f(z) = \dfrac{(z - \frac{1}{3})}{z(1 - \frac{1}{3}z)}$.

b) Show that if f and g are coverings of positive degree, then so is $f.g$.

c) Let $a \in \mathbf{R}$ with $\frac{1}{2} < a < 1$, and let

$$f(z) = z^3 \frac{1 - az}{z - a}.$$

By considering $\dfrac{zf'(z)}{f(z)}$ at $z = \pm 1$ or otherwise, show that f is not a covering.

[20 marks]

3. a) If a degree one homeomorphism f of the circle has at least one periodic point, state what restrictions there are on the periods of any periodic points of f. Find all periodic points of the following degree one homeomorphisms. In each example, give, without proof, all possibilities for the set $\overline{\{f^n(z) : n > 0\}}$ for varying $z \in S^1$.

(i) $f(z) = \dfrac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$,

(ii) $f(z) = \dfrac{\frac{1}{2} - z}{1 - \frac{1}{2}z}$ (possibly by considering $f \circ f$),

(iii) $f(z) = e^{2i\pi^2} z$.

b) Explain what it means for a fixed point $z = e^{2\pi i\theta}$ of a continuous circle map f to be *positively* or *negatively semistable*. You should give the definition in terms of a lift $\widetilde{f}$ of f such that $\widetilde{f}(\theta) = \theta$.

Let $f : S^1 \to S^1$ be a degree one homeomorphism with exactly one fixed point $z = e^{2\pi i\theta}$. By considering $\widetilde{f}(t) - t$ for a lift $\widetilde{f}$ fixing θ or otherwise, show that z is semistable. [20 marks]

4. a) Define what is meant by a *semiconjugacy* $\varphi : S^1 \to S^1$ between continuous maps f, $g : S^1 \to S^1$, and what is meant by a *conjugacy*. Let f be of degree d with $|d| > 1$ and let f have a continuously differentiable lift $\widetilde{f}$. Give, in terms of the derivative $\widetilde{f}'$, a condition for f to be conjugate to $g : z \mapsto z^d$.

b) Let d be an odd integer, with $|d| > 1$. Let f have a lift $\widetilde{f}$, where

$$\widetilde{f}(\theta) = d\theta + \frac{a}{2\pi}(\sin 2\pi\theta + (\sin 2\pi\theta)^3).$$

By considering where the function $h(x) = x(4 - 3x^2)$ has its maximum and minimum on the interval $[-1, 1]$ or otherwise, show that f is conjugate to $g : z \mapsto z^d$ when

$$|a| \leq \frac{9}{16}(|d| - 1).$$

By computing $\widetilde{f}'(0)$ and $\widetilde{f}'(\frac{1}{2})$ or otherwise, show that f and g are not conjugate if

$$|d| - 1 < |a| < |d| + 1.$$

c) Now let f be defined as in b), but with $d = 1$ and $0 < |a| < 9/16$. Show that one of ± 1 is a stable fixed point of f. Suppose that $\varphi : S^1 \to S^1$ is continuous and $\varphi(f(z)) = \varphi(z)$ for all $z \in S^1$. Show that $\varphi(f^n(z)) = \varphi(z)$ for all integers $n \geq 1$. Hence or otherwise, show that $\varphi(z) = \varphi(\pm 1)$ for all $z \in S^1$, quoting any results that you use. You may assume that f is a homeomorphism.

[20 marks]

5. a) Let $\{a_j\}_{j\geq 1}$ be one of the sequences

$$0^n(01)^\infty,\ (01)^n0^\infty,\ (10)^n0^\infty\,(\ n\geq 0).$$

In each of the three cases, find θ, where

$$\theta=\sum_{j=1}^{\infty}\frac{a_j}{2^j}. \tag{1}$$

We say that a sequence $\{a_j\}_{j\geq 1}$ of 0's and 1's is a *symbolic sequence* for $z\in S^1$ if $z=e^{2\pi i\theta}$ for $\theta\in[0,1]$ and (1) holds. Show that if $t,\ s\in[0,1]$, and $e^{2\pi it}$ and $e^{2\pi is}$ have symbolic sequences $\{a_j\}_{j\geq 1}$ and $\{b_j\}_{j\geq 1}$ with $a_j=b_j$ for $1\leq j\leq N$, then $|t-s|\leq 2^{-N}$.

b) Now let $f(z)=z^2$, let $\{a_j\}_{\geq 1}$ be a symbolic sequence for $z\in S^1$, and let n be any integer ≥ 1. Give a symbolic sequence for $f^n(z)$.

Now let z have symbolic sequence

$$0^1(01)^1 0^2(01)^2 0^3(01)^3\cdots$$

Let N be a positive integer. Show that if n is sufficiently large, then the symbolic sequence for $f^n(z)$ starts with

$$0^k(01)^N \text{ or } (01)^k0^N \text{ or } (10)^k0^N, \text{ for some } 0\leq k\leq N.$$

Show that there are sequences $\{n_r\}$ and $\{m_r\}$ of positive integers such that

$$\lim_{r\to\infty} f^{n_r}(z)=1,\ \lim_{r\to\infty} f^{m_r}(z)=e^{2\pi i/3}.$$

You may assume that if $t,\ s\in[0,1]$ with $|2\pi(t-s)|<1$, then

$$|e^{2\pi it}-e^{2\pi is}|\leq 4\pi|t-s|.$$

[20 marks]

6.

a) Define the *rotation number* $\rho(\widetilde{f})$ of a continuous increasing map $\widetilde{f}:\mathbf{R}\to\mathbf{R}$ satisfying $\widetilde{f}(\theta+1)=\widetilde{f}(\theta)+1$ for all θ, quoting any result that you use. Then define $\rho(f)$ for any continuous monotone degree one circle map f. Let there be $x_0\in\mathbf{R}$ and integers p and q with $q>0$ such that

$$\widetilde{f}^q(x_0)=x_0+p.$$

Show that $\rho(\widetilde{f})=p/q$.

b) Now let $|\lambda|=1$, and let $f_\lambda:S^1\to S^1$ be the homeomorphism

$$f_\lambda(z)=\lambda z^2\frac{z-4}{1-4z}.$$

Let $\widetilde{f}$ be a lift of f_1. Show that $\widetilde{f}_a$ is a lift of f_λ, where

$$\widetilde{f}_a(\theta)=\widetilde{f}(\theta)+a,\ \lambda=e^{2\pi ia}.$$

Compute $\rho(f_1)$ and $\rho(f_{-1})$, possibly by first computing $f_1(1)$ and $f_{-1}^2(1)$. Also, compute $f_1'(1)$ and $(f_{-1}^2)'(1)$. State what this implies about the point $1\in S^1$ in each case.

c) Let f_λ, $\widetilde{f}_a$ be as in b), and let

$$F(\theta,a)=\widetilde{f}_a(\theta)-\theta,\ G(\theta,a)=\widetilde{f}_a^2(\theta)-\theta-1.$$

Show that $\theta\mapsto F(\theta,a)$ takes both positive and negative values for a sufficiently small, and similarly for $\theta\mapsto G(\theta,a)$ if a is sufficiently near $\frac{1}{2}$. State what this implies for $\rho(f_\lambda)$ for λ sufficiently near 1 or -1.

[20 marks]

7.

a) State Denjoy's Theorem for a continuous degree one monotone map $f : S^1 \to S^1$ with lift $\widetilde{f} : \mathbf{R} \to \mathbf{R}$.

b) In both the following examples, $f : S^1 \to S^1$ is given by a lift $\widetilde{f}$. In each example, sketch the graph of $\widetilde{f}$ for some values of a, $b \in \mathbf{R}$, $|a| < 1$. For each example, determine for which values of a and $\rho(\widetilde{f})$ the conditions of Denjoy's Theorem are satisfied. Give brief reasons for your answers.

(i) $\widetilde{f}(\theta) = \theta + b + \dfrac{a}{2\pi} \cos 2\pi\theta$.

(ii) $\widetilde{f}(\theta) = \theta + b + \dfrac{a}{2\pi} \text{Max}(\sin 2\pi\theta, 0)$.

c) Let $f : S^1 \to S^1$ satisfy the conditions of Denjoy's Theorem, with $\rho(f) = \alpha \bmod \mathbf{Z}$. Describe without proof the set

$$\overline{\{e^{2\pi i n\alpha} z : n \in \mathbf{Z},\ n \geq 0\}}.$$

Then explain how Denjoy's Theorem can be applied to describe the set

$$\overline{\{f^n(z) : n \in \mathbf{Z},\ n \geq 0\}}$$

for any $z \in S^1$.

[20 marks]

8. a) Let $\alpha \in \mathbf{R}$ be irrational. Show that if $N > 0$ is an integer, then there are coprime integers p and q with $0 < q \leq N$ such that

$$|q\alpha - p| < 1/q. \tag{1}$$

Now let $\alpha = \sqrt{2}$. By considering the n'th power of the equation

$$(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$$

or otherwise, show that there are integers p_n and q_n with

$$|q_n\sqrt{2} - p_n| < \frac{1}{q_n\sqrt{2}},$$

where

$$\sqrt{2}q_n = \tfrac{1}{2}(1 + \sqrt{2})^n - \tfrac{1}{2}(1 - \sqrt{2})^n.$$

b) Deduce from (1) that, for each integer j with $0 \leq j < q$, there is an integer r, $0 \leq r < q$, and $n \in \mathbf{Z}$, such that

$$\left| \; r\alpha - n - \frac{j}{q} \; \right| \; < \frac{1}{q}.$$

Show that for such r, j and q,

$$q \int_{j/q}^{(j+1)/q} |\sin 2\pi r\alpha - \sin 2\pi t| dt \leq \int_{(j-1)/q}^{(j+1)/q} |2\pi \cos 2\pi u| du.$$

[*Hint*: it may be helpful to write $\sin 2\pi r\alpha - \sin 2\pi t$ as the integral of some function over a short interval.]

Hence, or otherwise, show that, for such q,

$$\left| \; \sum_{r=0}^{q-1} \sin 2\pi r\alpha \; \right| \; \leq 8.$$

[*Hint*: compute

$$q \int_{j/q}^{(j+1)/q} dt.]$$