SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist.

a) $x_n = (-1)^n \sqrt{n}$ $(n \geq 0)$.

b) $x_n = (n+2)/(n+1)$ $(n \geq 0)$. [8 marks]

2. Let the continued fraction expansion of $\sqrt[3]{3} = 1.44224957\ldots$ be given by $[a_0, a_1, a_2, a_3, \ldots]$. Using your calculator, determine a_n for $0 \leq n \leq 3$ (you do not need to write anything down other than the values of each a_n). Hence calculate the first 4 convergents to $\sqrt[3]{3}$. (Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$).

[7 marks]

3. Let $f\colon [0,1] \to [0,1]$ be a map which has a continuous derivative $f'(x)$. What does it mean for a fixed point p of f to be stable? State how the derivative $f'(p)$ can be used to determine whether p is stable, and sketch a spider diagram near a stable fixed point to illustrate your answer.

For each $r \in [0,4]$, let $f_r\colon [0,1] \to [0,1]$ be given by $f_r(x) = rx(1-x)$. Determine the values of r for which the fixed point $p = 0$ of f_r is stable.

[7 marks]

4. State without proof whether each of the following sets is open, closed, both, or neither.

a) $(0,1)$, as a subset of $\mathbf{R}$.

b) $\{(x,0) : x \in (0,1)\}$, as a subset of $\mathbf{R}^2$.

c) $\{(x,y) : 1 \leq x^2 + y^2 \leq 2\}$, as a subset of $\mathbf{R}^2$.

d) $\{(x,y) : 1 \leq x^2 + y^2 < 2\}$, as a subset of $\mathbf{R}^2$.

[8 marks]

5. A student spends each hour of her life either sleeping or drinking. If she is sleeping in a given hour, she will sleep the next hour with probability 2/3, and drink with probability 1/3. If she is drinking, then she will sleep the next hour with probability 1/2, and drink with probability 1/2.

Write down the matrix of transition probabilities which governs her behaviour. In the long term, what proportion of her life does she spend on each activity?

[7 marks]

6. Calculate the Fourier series expansion of $|t|$ $(t \in [-\pi, \pi))$. [10 marks]

7. The Fourier series expansion of t^2 $(t \in [-\pi, \pi))$ is

$$\frac{\pi^2}{3} + \sum_{r=1}^{\infty} \frac{4(-1)^r}{r^2} \cos rt.$$

Using Parseval's theorem, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}.$$

[8 marks]

SECTION B

8. Let

$$f(x) = x + 1 - \frac{x^2}{6}.$$

Let the sequence (x_n) be defined iteratively by $x_0 = 2$ and $x_{n+1} = f(x_n)$ for each $n \geq 0$. Prove that (x_n) is an increasing sequence which tends to $\sqrt{6}$ as $n \to \infty$. (You may use any results from the lectures without proof, but they should be clearly stated).

Show that $|x_n - \sqrt{6}| < (1/3)^n$. How large should n be in order to ensure that x_n agrees with $\sqrt{6}$ to 50 decimal places? [15 marks]

9. What does it mean for an infinite set S to be countable?

a) Show that $\mathbf{Q}$ is countable.

b) Show that $\mathbf{R}$ is uncountable.

c) Show that if S, T, and U are all countable infinite sets, then their union $S \cup T \cup U$ is also countable.

[15 marks]

10. State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f\colon [0,1] \to [0,1]$).

Determine the Markov graphs of the two period 5 patterns $(1\,2\,3\,4\,5)$ and $(1\,3\,4\,2\,5)$. Show that a continuous map $f\colon [0,1] \to [0,1]$ with a periodic orbit of pattern $(1\,2\,3\,4\,5)$ must have a period 3 orbit, while one with a periodic orbit of pattern $(1\,3\,4\,2\,5)$ need not have. What other periods of orbits must there be in each of the two cases? [15 marks]

11. Let $f\colon [0,1] \to [0,1]$ be given by $f(x) = 4x(1-x)$. Show that if $x \in [0,1]$, then $f^n(x) = (\sin(2^n\theta))^2$, where $\theta \in [0, \pi/2]$ is such that $x = (\sin\theta)^2$.

Hence, or otherwise, show that f^n has 2^n fixed points for each $n \geq 1$.

Suppose that n is a prime number. How many period n orbits does f have? Deduce that $2^n - 2$ is divisible by n for all prime numbers n. [15 marks]

12.a) Calculate the Fourier series expansion of t $(t \in [-\pi, \pi))$.

b) The Fourier series expansion of t^4 $(t \in [-\pi, \pi))$ is

$$\frac{\pi^4}{5} + 8\sum_{r=1}^{\infty} \frac{(-1)^r(\pi^2 r^2 - 6)}{r^4} \cos rt.$$

Integrate this series term by term and use your result from part a) to determine the Fourier series expansion of t^5 $(t \in [-\pi, \pi))$.

c) Under what conditions is term by term differentiation of a Fourier series expansion valid?

[15 marks]

13. What does it mean for a function $f: \mathbf{R} \to \mathbf{R}$ to be odd? Explain why an odd periodic function has no constant or cosine terms in its Fourier series expansion.

Let

$$f(t) = \begin{cases} -1 & \text{if } -\pi \leq t < 0 \\ 1 & \text{if } 0 < t < \pi \end{cases}$$

(and $f(0) = 0$). Sketch the 2π-periodic extension of $f(t)$, and determine its Fourier series expansion.

By applying the Fourier series theorem at $t = \pi/2$, show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \frac{\pi}{4}.$$

[15 marks]