SECTION A

1. State (without proof) whether or not each of the following sequences (x_n) is i) increasing; ii) decreasing; iii) bounded above; iv) bounded below. State also the supremum, infimum, maximum, and minimum of each sequence for which they exist.

a) $x_n = 1 - \frac{1}{n+1}$ $(n \geq 0)$.

b) $x_n = (n-1)^2/2$ $(n \geq 0)$. [8 marks]

2. Calculate the value of, and the first four convergents to, each of the following continued fractions:

a) $[2, 2, 2, 2, 2, 2 \ldots]$.

b) $[2, 1, 2, 1, 2, 1, \ldots]$.

(Recall the formulae: $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 2$; $q_0 = 1$, $q_1 = a_1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 2$).

[8 marks]

3. Let $f_r : \mathbf{R} \to \mathbf{R}$ be given by $f_r(x) = r - x^2$, where the parameter r can take any real value. Calculate the fixed points of f_r, and determine the range of values of r for which each is stable. (You are not required to determine the stability of the fixed points when $f_r'(x)$ is ± 1). [7 marks]

4. State without proof whether each of the following sets is open, closed, both, or neither.

a) $(0, 1)$, as a subset of $\mathbf{R}$.

b) $\mathbf{R}$, as a subset of $\mathbf{R}^2$.

c) $\{(x, y) : 0 < x^2 + y^2 \leq 1\}$, as a subset of $\mathbf{R}^2$.

d) $\{(x, y) : 3x - y \leq 3\}$, as a subset of $\mathbf{R}^2$.

[8 marks]

5. A map $f: [0, 1] \to [0, 1]$ is such that f has 1 fixed point, f^2 has 5 fixed points, f^3 has 10 fixed points, and f^4 has 25 fixed points. How many periodic orbits does f have of each of the periods 2, 3, and 4?

[6 marks]

6. Calculate the Fourier series expansion of $|t|$ ($t \in [-\pi, \pi)$). [10 marks]

7. Sketch the 2π-periodic extensions of each of the following functions $f(t)$, defined for $t \in [-\pi, \pi)$. State whether each is i) continuous, ii) piecewise continuous, iii) differentiable, and iv) piecewise differentiable. Give the values of $f(t^-)$ and $f(t^+)$ at each discontinuity.

a) $f(t) = t^2$.

b)

$$f(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ t & \text{if } t < 0. \end{cases}$$

[8 marks]

SECTION B

8. Using the Newton-Raphson formula, construct an iteratively defined sequence (x_n), starting with $x_0 = 2$, which tends to $\sqrt[3]{7}$ as $n \to \infty$. Prove that the sequence does indeed tend to $\sqrt[3]{7}$. (You may use any results from the lectures without proof, but they should be clearly stated). [15 marks]

9. What does it mean for an infinite set S to be countable?

a) Show that $\mathbf{Q}$ is countable.

b) Show that $\mathbf{R}$ is uncountable.

c) Show that if S, T, and U are all countable infinite sets, then their union $S \cup T \cup U$ is also countable.

[15 marks]

10. State Sharkovsky's theorem (concerning the possible sets of periods of continuous maps $f\colon [0,1] \to [0,1]$).

Determine the Markov graphs of the two period 5 patterns (1 4 3 2 5) and (1 5 3 2 4). Show that a continuous map $f\colon [0,1] \to [0,1]$ with a periodic orbit of pattern (1 4 3 2 5) must have a period 3 orbit, while one with a periodic orbit of pattern (1 5 3 2 4) need not have. What other periods of orbits must there be in each of the two cases? [15 marks]

11. What does it mean for a square matrix P to be a stochastic matrix? Under what conditions is it true that there is a unique eigenvector $\mathbf{x}$ of P with eigenvalue 1, such that $P^n\mathbf{x}_0 \to \mathbf{x}$ as $n \to \infty$ for all vectors $\mathbf{x}_0$ with non-negative entries adding up to 1?

A student has three courses to revise for her resits: Iteration and Fourier Series; Complex Functions; and Rings, Fields, and Combinatorics. She has a very long time for her revision and decides that, to make it more interesting, she will toss a fair coin at the end of every minute and change the course she is revising if it comes up heads.

If it comes up heads when she is revising Iteration and Fourier Series, then she is equally likely to switch to either of the other two courses; whereas if it comes up heads when she is revising one of the other two courses, she is twice as likely to switch to Iteration and Fourier series as to the other possibility.

In the long run, what proportion of her time will be spent on Iteration and Fourier Series?

[15 marks]

12.

Calculate the Fourier Series expansion of

$$f(t) = \begin{cases} 0 & \text{if } -\pi \le t < 0 \\ t & \text{if } 0 \le t < \pi. \end{cases}$$

By applying the Fourier Series theorem at $t = 0$, show that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

[15 marks]

13. Calculate the Fourier series expansion of t^2 ($t \in [-\pi, \pi)$). By applying Parseval's theorem, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}.$$

[15 marks]