0.4pt0in

PAPER CODE NO.
2MM16

THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 1997 EXAMINATIONS

Degree of Bachelor of Science : Year 1

MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55 % of the available marks. The vectors **i**,**j**,**k** are the unit vectors in the directions of the coordinate axes Ox, Oy, and Oz respectively.

THE UNIVERSITY
of LIVERPOOL

SECTION A

1. For the matrices **A** and **B**, where

$$\mathbf{A} = \begin{pmatrix} 12 & 0 & 2 \\ 3 & 1 & -4 \\ -2 & 3 & N \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

where N is a constant find the matrix **AB**.

[6 marks]

2. Find the magnitude of the vector $4\mathbf{a} - \mathbf{b}$, where

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

[5 marks]

3. The vectors **p** and **q** are given by

$$\mathbf{p} = \mathbf{i} - \mathbf{j} + \mathbf{k} \text{ and } \mathbf{q} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Find (i) the scalar product $\mathbf{p}.\mathbf{q}$, (ii) the vector product $\mathbf{p} \times \mathbf{q}$.

[5 marks]

4. The position vector of a particle of at time t is

$$\mathbf{r}(t) = 3\mathbf{i} + (2 - 3t)\mathbf{j} + (29.6 - 4.9t^2)\mathbf{k}\ m$$

show that the particle moves in the plane $\mathbf{r}.\mathbf{i} = p$ where the value of p should be stated. Find the particle's speed at time $t = 3s$, correct to 4 significant figures. [5 marks]

5. The force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ N moves a particle of mass 3 kg in the straight line from the point with coordinates $(1, -2, 4)$ m to the point with coordinates $(7, 6, 5)$ m.
Find the work done by this force. [5 marks]

6. Given that the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + a\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$ are orthogonal find the value of a. [4 marks]

Show that (i) $\mathrm{div}(r^3\mathbf{r}) = 6r^3$ and (ii) $\mathrm{curl}(r^3\mathbf{r}) = \mathbf{0}$
[You may use $\frac{\partial r}{\partial s} = \frac{s}{r}$ where $s = x$ or y or z.]

[4 marks]

8. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 6x.$$

Express your solution in the form $y = f(x)$.

[6 marks]

9. Solve the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 7\,\frac{\mathrm{d}y}{\mathrm{d}t} + 12\,y = 0.$$

given that $y = 0$ and $\mathrm{d}y/\mathrm{d}t = 1$ when $t = 0$.
What is the value of y when $t = 0.1$? Give your answer to 3 decimal places.

[7 marks]

10. Show that, for n a positive integer

$$\int_0^{\pi} \theta\cos(n\theta)\mathrm{d}\theta = \frac{1}{n^2}((-1)^n - 1).$$

Write down

$$\int_{-\pi}^{0} -\theta\cos(n\theta)\mathrm{d}\theta$$

Hence, show that the Fourier series for the periodic function

$$f(\theta) = |\theta|, \quad (-\pi < \theta < \pi),$$

is

$$f(\theta) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(\theta) + \frac{1}{9}\cos(3\theta) + \frac{1}{25}\cos(5\theta) + \cdots\right).$$

[8 marks]

SECTION B

11. Using Gaussian elimination, reduce the matrix

$$\begin{pmatrix} 1 & -2 & 3 & 5 \\ 2 & -5 & 6 & 3 \\ 1 & -3 & -2 & 3 \end{pmatrix}$$

to the form

$$\begin{pmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{pmatrix}$$

where the ?'s denote numbers which have to be determined.
Hence, or otherwise, find the solutions of the linear equations

$$\begin{aligned} x - 2y + 3z &= 5 \\ 2x - 5y + 6z &= 3 \\ x - 3y - 2z &= 3 \end{aligned}$$

and check your results. [15 marks]

12. (a) Find the scalar triple product $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ for the vectors:

$$\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = 5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}, \quad \mathbf{c} = 2\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}.$$

[8 marks]

(b) At time t, the position vector of a particle moving in a plane is $\mathbf{r} = r(\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}) = r\hat{\mathbf{r}}$, where $r = |\mathbf{r}|$. Given that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\underline{\hat{\theta}} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\underline{\hat{\theta}} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\mathbf{r}}$$

find the particle's acceleration, expressing your answer in terms of the unit vectors $\hat{\mathbf{r}}$ and $\underline{\hat{\theta}}$.

[7 marks]

13. At time t, a particle of mass 4 kg has position vector,

$$\mathbf{r}(t) = 2\sin\left(\frac{\pi t}{2}\right)\mathbf{i} + 3\cos\left(\frac{\pi t}{2}\right)\mathbf{j} \text{ m}.$$

Find its distance from the origin at times (i) $t = 2$, (ii) $t = 3$.
Find the velocity $\mathbf{v}$ and acceleration at time t. Find the force $\mathbf{F}$ which gives rise to this acceleration. Show that

$$\int_0^3 \mathbf{F}.\mathbf{v}\mathrm{d}t = \frac{5\pi^2}{2} \text{ J}.$$

[15 marks]

14. The magnetic field $\mathbf{B}$ at the point with position vector $\mathbf{r}$ due to a current I in a long straight wire along the y- axis is given by $\mathbf{B} = \text{curl } \mathbf{A}$, where

$$\mathbf{A} = -\frac{\mu_0 I}{4\pi}\ln\left(\frac{x^2+z^2}{a^2}\right)\mathbf{j}$$

where a is a constant.
Find $\mathbf{B}$ and show that div $\mathbf{B} = 0$.

[15 marks]

15. (a). The displacement $x(t)$ of a particle satisfies the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3\,x = 6.$$

The original displacement is 0 m and the initial speed is 0 ms^{-1}.
Find the displacement at time t. Sketch $x(t)$. [9 marks]

(b). Find values of a and b given that the polynomial $P(x) = ax^2 + b$ satisfies the Legendre differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and the normalisation condition

$$\int_{-1}^{1} P^2(x)\mathrm{d}x = \frac{2}{5}.$$

[6 marks]

0.4pt0in

16. (a). Determine whether the following funtions, each with period 2π, will have only cosine, sine or both cosine and sine terms in their Fourier series:

(i) $f(\theta) = 4\theta$ for the interval $-\pi < \theta < \pi$,
(ii) $f(\theta) = \theta^3 \sin(\theta)$ for the interval $-\pi < \theta < \pi$,
(iii) $f(\theta) = \theta^2 \sin(3\theta)$ for the interval $-\pi < \theta < \pi$,
(iv) $f(\theta) = \theta^4 \cos(2\theta)$ for the interval $-\pi < \theta < \pi$,
(v) $f(\theta) = (1+\theta)\cos(\theta)$ for the interval $-\pi < \theta < \pi$,
(vi) $f(\theta) = -\theta - \pi$ for the interval $-\pi < \theta < 0$, $f(\theta) = -\theta + \pi$ for the interval $0 < \theta < \pi$.

[6 marks]

(b). Find the first three terms in the Fourier Series of the waveform given by the function $f(\theta) = 4\theta$ over the interval $-\pi < \theta < \pi$ if the period is 2π.
Sketch the waveform for $-2\pi \leq \theta < 2\pi$.

[9 marks]