SECTION A

1. For the matrices $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 11 & -2 & -1 \\ 3 & 10 & -5 \\ 7 & -2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

find the matrix $\mathbf{AB}=\mathbf{C}$. By comparing each of the columns of $\mathbf{B}$ with the corresponding column of $\mathbf{C}$, find one eigenvector of $\mathbf{A}$.

[6 marks]

2. Find the magnitude of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - 2\mathbf{b}$, where

$$\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k} \text{ and } \mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}.$$

[6 marks]

3. Find the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$, where

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \text{ and } \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

Hence, find the angle in degrees between these vectors. Give your answer to three decimal places.

[5 marks]

4. At time t, a particle of mass 3 kg has position vector

$$\mathbf{r}(t) = 4\sin(3t)\mathbf{i} + 3\cos(2t)\mathbf{j} \text{ m}.$$

Find the particle's velocity and acceleration at time t. Find the force which produces this acceleration.

[5 marks]

5. Find the work done by the force $\mathbf{F} = \mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$ N in moving a particle in a straight line from the point with coordinates $(1, 3, 1)$ m to the point with coordinates $(3, -2, -1)$ m.

[3 marks]

6. Find grad ϕ for the cases (i) $\phi = |\mathbf{r}|$ and (ii) $\phi = |\mathbf{r}|^{-2}$.

[4 marks]

7. Find (i) $\mathrm{div}(\mathbf{a} \times \mathbf{r})$ and (ii) $\mathrm{curl}(\mathbf{a} \times \mathbf{r})$, where $\mathbf{a}$ is a constant vector.

[5 marks]

8. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{y}{x} = \frac{\sin(x)}{x^2}.$$

Express your solution in the form $y = f(x)$.

[6 marks]

9. Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 5\,\frac{\mathrm{d}y}{\mathrm{d}t} + 4\,y = 4.$$

given that $y = 0$ and $\mathrm{d}y/\mathrm{d}t = 0$ when $t = 0$.
What is the value of y when $t = 1$? Give your answer to 3 decimal places.

[8 marks]

10. (i) The periodic function $f(\theta)$ is defined by

$$f(\theta) = \begin{cases} 0 & -\pi < \theta \leq 0 \\ \theta & 0 < \theta < \pi \end{cases} \quad \text{and} \quad f(\theta + 2\pi) = f(\theta)$$

Sketch $f(\theta)$ for $-4\pi < \theta < 4\pi$.

(ii) Given that the Fourier series for the periodic function

$$g(\theta) = |\theta|, \quad -\pi < \theta < \pi, \quad g(\theta + 2\pi) = g(\theta)$$

is

$$g(\theta) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(\theta) + \frac{1}{9}\cos(3\theta) + \frac{1}{25}\cos(5\theta) + \cdots\right).$$

Choose a value for θ to find the sum to infinity of the series

$$S_n = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2}.$$

[7 marks]

SECTION B

11. Row reduce the matrix

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 1 & k+3 & 2 \\ 4 & k+2 & 2 & 3 \end{bmatrix},$$

Hence, or otherwise, find the values of k for which the system of equations

$$\begin{aligned} 2x &+ y &+ 2z &= 1 \\ 3x &+ y &+ (k+3)z &= 2 \\ 4x &+ (k+2)y &+ 2z &= 3. \end{aligned}$$

has (i) no solutions, (ii) an infinite number of solutions. Find the solution when $k = 2$.

[15 marks]

12. (i) Find the scalar triple product $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ for the vectors:

$$\mathbf{a} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \mathbf{c} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}.$$

What geometrical property can your result represent? [7 marks]

(ii) For polar coordinates r, θ the radial and transverse components of the velocity of a particle which moves in a plane are $\dot{r}$ and $r\dot{\theta}$ respectively, and the corresponding components of the acceleration are $\ddot{r} - r\dot{\theta}^2$ and $2\dot{r}\dot{\theta} + r\ddot{\theta}$.
A particle describes the spiral $r = ae^{\theta}$, where a is constant, about the origin O. When at a distance r from O its radial velocity is h/r, where h is a constant. Prove that the acceleration of the particle is $2h^2/r^3$ directed towards O.

[8 marks]

13. At time t, a particle of mass 2 kg has position vector,

$$\mathbf{r}(t) = 3\,\cos(\pi t)\mathbf{i} + 12\,\sin(\pi t)\mathbf{j} + 4t\mathbf{k}\ \text{m}.$$

Find its distance from the origin at times (i) $t = 1$, (ii) $t = 5/2$.
Find the velocity $\mathbf{v}$ and acceleration at time t. Find the force $\mathbf{F}$ which gives rise to this acceleration. Show that

$$\int_1^{5/2} \mathbf{F}.\mathbf{v}\mathrm{d}t = -135\pi^2.$$

[15 marks]

14. The vector field $\mathbf{A}$ is given by

$$\mathbf{A} = -2xz^2\mathbf{i} + 3y^2z\mathbf{j} + (y^3 - 2x^2z)\mathbf{k}$$

Find $\mathrm{div}(\mathbf{A})$ and show that $\mathrm{curl}(\mathbf{A}) = 0$.
The vanishing of the curl shows that $\mathbf{A}$ is the gradient of a potential function ϕ.
Write down the formula for $\mathrm{grad}(\phi)$ in rectangular cartesian coordinates and use this formula to find ϕ as a function of x, y and z.

[15 marks]

15. (a). The charge $q(t)$ in a particular electrical circuit satisfies the differential equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + 8\frac{\mathrm{d}q}{\mathrm{d}t} + 15\,q = 30.$$

The current in the circuit is $I(t) = \mathrm{d}q/\mathrm{d}t$.
Given that $q(0) = 0$ and $I(0) = 0$, find $q(t)$ and $I(t)$.
Show that the current has a maximum value when $t = \frac{1}{2}\ln\frac{5}{3}$
Sketch $q(t)$ and $I(t)$.

[9 marks]

(b). Find values of a and b given that the polynomial $P(x) = ax^2 + b$ satisfies the Legendre differential equation

$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$

and $P(1) = 1$.

[6 marks]

16. Show that for $n \neq 0$

$$\int \theta^2 \cos(n\theta) \mathrm{d}\theta = \frac{n^2\,\theta^2 \sin(n\,\theta) + 2\,n\,\theta \cos(n\,\theta) - 2\sin(n\,\theta)}{n^3},$$

assuming that the constant of integration is zero.
Find the first four terms in the Fourier series of the waveform given by the function $f(\theta) = \theta^2$ over the interval $-\pi < \theta < \pi$ if the period is 2π.
Write down a formula for the term containing $\cos(n\theta)$.
Sketch the waveform for $-3\pi \leq \theta < 3\pi$.

[15 marks]