SECTION A

1. Let U be the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of digits,

$A = \{x \in U \mid x \text{ is even}\}$,

$S = \{x \in U \mid x = y^2 \text{ for some } y \in U\}$,

$T = \{0, 1, 4, 5, 8, 9\}$.

List the elements of each of the sets S, $A \cap T$, $T \cup A$, $T - S$, $U - (A \cup S)$.

[5 marks]

2. (i) Write down the number of strings of length 5 in the 4 letters a, b, c, d. Say how many of these strings

(ii) contain the letter a just once;

(iii) do not contain the letter b;

(iv) satisfy both (ii) and (iii).

[7 marks]

3. The government of Ruritania operates a National Lottery. Each ticket carries four numbers, all different, which may be chosen between 1 and 20 inclusive. How many possible such choices are there?

The rules of the Lottery have recently been changed: one of the four numbers is now designated as a 'bonus number'. How many essentially different tickets are there now?

[6 marks]

4. Write out rows 0 to 5 of Pascal's triangle. Expand $(x - 3y)^3$ and $(x^2 + 2)^5$ in powers of x.

[7 marks]

5. Calculate truth tables for the following expressions:

(i) $(P \to \overline{Q}) \vee (P \wedge \overline{Q})$;

(ii) $(P \vee Q) \to (\overline{P} \vee Q)$;

(iii) $(P \to Q) \vee (P \to \overline{Q})$.

Decide which, if any, is a tautology or contradiction.

[8 marks]

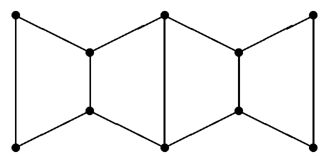
6. Decide whether or not the following arguments are valid, and give reasons for your decision:

(i) $P \to \overline{Q},\ Q \to \overline{R} \vdash P \to (\overline{Q} \vee \overline{R})$,

(ii) $P \to Q,\ P \to R \vdash Q \vee R$.

[8 marks]

7. For the graph G shown, decide



(i) how many vertices have valence 3?

(ii) whether or not the graph is simple;

(iii) the length of the shortest simple closed path.

(iv) Give also a spanning tree of the graph. [6 marks]

8. Say what is meant by the term 'partial order'. Explain why the relation R defined on the set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ by:

$$x\,R\,y \quad \text{if } \tfrac{x}{y} \text{ is a whole number}$$

is a partial order. Draw the Hasse diagram of R. [8 marks]

SECTION B

9. (a) Find out how many strings of length 5 in the alphabet $\{0, 1, 2, 3\}$ satisfy

(i) they do not start with 0,

(ii) 2 appears just twice,

(iii) no two consecutive digits are the same.

(b) The sequence $\{c_n\}$ is defined recursively by $c_0 = 0$, $c_1 = 1$ and, for $n \geq 2$, $c_n = 5c_{n-1} - 4c_{n-2}$. Prove by induction on n that for $n \geq 0$, $c_n = (4^n - 1)/3$.
[15 marks]

10.(a) The sets $A \cap B$, A and B have 6, 10 and 12 elements respectively: what is $|A \cup B|$? Given also that $|A \cap C| = 4$, $|B \cap C| = 8$, $|C| = 11$ and $|A \cup B \cup C| = 18$, determine $|A \cap B \cap C|$.

(b) Obtain the disjunctive normal forms of

(i) $\overline{(Q \vee P)} \wedge (Q \to R)$,

(ii) $(\overline{R} \wedge P) \wedge \overline{(Q \to R)}$. [15 marks]

11.(a) The variables are natural numbers 1,2,3, ...; the statement $e(n)$ means that n is even; $d(n)$ means that the decimal expression of n ends with 0; $s(n)$ means that $n < 10$. Decide which of the following are true:

(i) $(\exists n)(d(n) \wedge s(n))$,

(ii) $(\forall n)(e(n) \rightarrow d(n))$,

(iii) $(\forall n)(d(n) \rightarrow e(n))$,

(iv) $(\exists n)((\forall x)e(n+x))$.

(v) $(\forall x)((\exists n)e(n+x))$.

(b) Express the following argument in the language of predicate logic, using x to stand for an animal, T for 'Tiger', $C(x)$ for "x is a cat", $G(x)$ for "x has green eyes", $D(x)$ for "x can see in the dark", $H(x)$ for "x hunts at night", and $F(x)$ for "x fears being hunted"; and hence prove that it is valid.

All cats have green eyes
Animals with green eyes can see in the dark
Animals can see in the dark only if they either hunt at night or fear being hunted
'Tiger' is a cat who fears nothing
So 'Tiger' hunts at night.

[15 marks]

12. Let Γ be a simple graph; let N_d be the number of vertices of valency d for each d. Show that the numbers V of vertices and E of edges are given by $V = \sum N_d$ and $2E = \sum dN_d$. Give also the relation between E and V which holds when Γ is a tree.

Show that if Γ is a tree all of whose vertices have valence 1 or 3, then $N_1 = N_3 + 2$. Find two non-isomorphic trees of this kind, each with 10 vertices.

[15 marks]

13. Say what is meant by the term 'equivalence relation'. If R is an equivalence relation on a set Z, say how Z is partitioned into equivalence classes for R.

Let PX denote the set of subsets of $X = \{A, B, C, D\}$. The relation E is defined on PX by:

$S_1 \, E \, S_2$ if S_1 and S_2 have the same number of elements.

Show that E is an equivalence relation. Say how many equivalence classes there are, and list the elements of PX in each of them. [15 marks]