SECTION A

1. Let U be the set $\{0,1,2,3,4,5,6,7,8,9\}$ of digits, $A = \{1,2,4,8\}$, $P = \{2,3,5,7\}$, $T = \{0,1,4,5,8,9\}$. List the elements of each of the sets $A \cap T$, $P \cup A$, $T - (A \cup P)$, $U - (P \cup T)$. One of these four sets is a subset of another: which?

[5 marks]

2. Write down the number of strings of length 6 in the 4 letters a, b, c, d. Say how many of these strings

(i) contain the letter a just once;

(ii) start with the letter b;

(iii) satisfy both (i) and (ii).

[7 marks]

3. The red cards are removed from a pack of playing cards, leaving 13 spades and 13 clubs. How many hands of 7 cards can be dealt from those that remain? How many of these contain 2 spades and 5 clubs? (You need not evaluate the binomial coefficients.)

[6 marks]

4. Write out rows 0 to 5 of Pascal's triangle. Expand $(x+x^{-1})^4$ and $(x-3y)^5$ in powers of x.

[7 marks]

5. Calculate truth tables for the following expressions:

(i) $(P \to Q) \wedge (P \wedge \overline{Q})$;

(ii) $(P \vee Q) \wedge (\overline{P} \vee Q)$;

(iii) $(P \to Q) \vee (P \wedge \overline{Q})$.

Decide which, if any, is a tautology or contradiction. [8 marks]

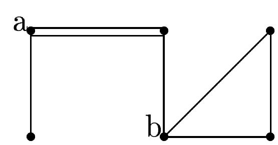
6. Decide whether or not the following arguments are valid, and give reasons for your decision:

(i) $P \to Q,\ \overline{R} \to \overline{Q} \vdash P \to R$,

(ii) $P \vee Q,\ P \to R \vdash Q \vee R$.

[8 marks]

7. For the graph G shown, decide



(i) the valencies of the vertices labelled a, b;

(ii) whether or not the graph is simple;

(iii) the length of the shortest path from a to b;

(iv) whether or not it is possible to remove one edge so that the resulting graph is disconnected.

Give also a spanning tree of the graph. [6 marks]

8. Say what is meant by the term 'equivalence relation'. The relation defined on the set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by:

$x\,R\,y$ if $x - y$ is even

is an equivalence relation. Say how many equivalence classes there are, and list the elements of U in each of them. [8 marks]

SECTION B

9. (a) Find out how many strings of length 5 in the alphabet $\{0, 1, 2, 3\}$ have

(i) 0 appearing just twice,

(ii) 2 appearing just once and 3 just once,

(iii) both (i) and (ii),

(iv) either (i) or (ii).

(b) How many ways are there of distributing 20 (identical) sheets of paper between 4 examinees, assuming that each gets at least two? [15 marks]

10.(a) Prove by induction on n that for $n \geq 1$,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

(b) Obtain the disjunctive normal forms of

(i) $\overline{(Q \to P)} \wedge (Q \to R)$,

(ii) $(R \vee P) \wedge \overline{(Q \vee R)}$. [15 marks]

11.(a) The variables are natural numbers 2,3, ... (excluding 1); the statement $e(n)$ means that n is even; $d(n)$ means that $n > 5$; $s(n)$ means that n is a power of 2 (i.e. $n = 2^k$ for some whole number k). Decide, giving reasons, which of the following are true:

(i) $(\exists n)(e(n) \wedge \overline{d(n)})$,

(ii) $(\forall n)(e(n) \to s(n))$,

(iii) $(\forall n)(s(n) \to e(n))$.

(b) Express the following argument in the language of predicate logic, using $C(x)$ for "x is a cat" , $D(x)$ for "x is a dog", $G(x)$ for "x is a good pet", and $E(x)$ for "x has green eyes", and hence prove that it is valid.

The only good pets are cats and dogs. All cats have green eyes. Max is a good pet who has brown eyes. So Max is a dog. [15 marks]

12. Let Γ be a simple graph; let N_d be the number of vertices of valency d for each d. Show that the numbers V of vertices and E of edges are given by $V = \sum N_d$ and $2E = \sum dN_d$. Give also the relation between E and V which holds when Γ is a tree.

Draw a simple graph with $N_2 = 4$, $N_3 = 2$ and $N_d = 0$ for other values of d. Find all spanning trees of your graph. How many are there? [15 marks]

13. Let S denote the set of strings ab of length 2 in the alphabet $A = \{1, 2, 3, 4\}$. The relations R_1, R_2 are defined on S by:

$ab\ R_1\ AB$ if $a \leq A$,

$ab\ R_2\ AB$ if $b \leq B$.

Show that these are partial orders, and that so is $R = R_1 \cap R_2$. Draw the Hasse diagram for R. [15 marks]