2MA66 Summer 1999

Instructions to candidates.

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be counted.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \quad , \quad R = R^{\mu}{}_{\mu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$c = 2.998 \times 10^{8} \text{ ms}^{-1}$$

1. State clearly one of Einstein's principles of special relativity.

An observer A is at rest in an inertial frame S whilst observer B moves relative to S at constant velocity v in the positive x direction in an inertial frame S'. S and S' are synchronised at time $t = 0$. At time $t = T$ in S, A sends a light signal to B who receives it at $t' = kT$ in S' and immediately reflects it back to A. Sketch these events on a spacetime diagram and show that

$$k(v) \;=\; \sqrt{(c+v)/(c-v)}\;\;.$$

If additionally an observer C moves in an inertial frame S'' at constant velocity u in the positive x direction relative to S', where S'' is also synchronised with S at $t = 0$, deduce the relation between $k(u)$, $k(v)$ and $k(w)$ where w is the speed of S'' relative to S. Find w in terms of u and v.

A particle P at rest at the origin is approached along the x-axis from $x = -\infty$ by a particle L travelling at constant speed $2c/5$. At time $t = 0$ P emits two particles Q and R. Particle Q travels in the positive x-direction from P at constant speed $4c/5$ whilst R travels in the opposite direction at speed $3c/5$. Determine the velocities of Q and R as observed by L.

2. Define the momentarily comoving reference frame, (MCRF). Give brief reasons why the definition of uniform acceleration in special relativity is based on the MCRF.

A rocket is located at the origin O of an inertial frame S and undergoes from rest a uniform proper acceleration of a in the positive x-direction. Determine its velocity as a function of t and the equation of the rocket's worldline in S.

At blast-off an observer at $x = 2c^2/a$ in S sends a light signal towards the rocket. Sketch both events on the same spacetime diagram. Show that according to the observer the light reaches the rocket at $t = 4c/(3a)$. What time does the clock on the rocket read when the light signal arrives?

[You may quote the formula $d(u\gamma(u))/dt$ for the proper acceleration.]

3. (a) A particle of mass M decays into three identical particles of mass m in such a way that the magnitudes of their momenta are equal. Using the laws of conservation of energy and momentum, compute the speed of one of the remnant particles. Clearly state with reasoning a condition on the masses M and m for the process to be possible.

(b) A particle of rest mass m and kinetic energy $3mc^2$ collides with a particle of rest mass $2m$ which is at rest. The two particles coalesce to form one particle of rest mass M. Compute M in terms of m and show that its resulting speed is $c\sqrt{5/12}$.

4. Consider a Cartesian coordinate system $x^\mu = (x, y)$ and a plane polar coordinate system $x^{\mu'} = (r, \theta)$ which are related by

$$x = r\cos\theta \qquad y = r\sin\theta$$

where the transformation matrix $\Lambda = \left(\Lambda^{\mu'}{}_{\mu}\right)$ is given by

$$\left(\Lambda^{\mu'}{}_{\mu}\right) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} .$$

Compute the matrix $\left(\Lambda^{\mu}{}_{\mu'}\right)$.

Let V^μ and $T^\mu{}_\nu$ be respectively rank 1 and 2 tensors. Write down the transformation rule for each tensor between coordinate systems x^μ and $x^{\mu'}$ and the definition of their covariant derivatives with respect to x^σ.

Let

$$V^\mu = \begin{pmatrix} x(x^2 - 3y^2) \\ y(3x^2 - y^2) \end{pmatrix}$$

and show that

$$V^r = r^3\cos 2\theta \quad , \quad V^\theta = r^2\sin 2\theta \ .$$

If the only non-zero Christoffel symbols for plane polar coordinates are

$$\Gamma^r_{\theta\theta} = -r \quad , \quad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$$

show that

$$V^r{}_{;\theta} = -3r^3\sin 2\theta \quad \text{and} \quad V^r{}_{;\theta r} = -6r^2\sin 2\theta \ .$$

Without further calculation state with reasons the value of $V^r{}_{;r\theta}$.

5. Demonstrate, using clearly stated symmetry properties, that for two dimensional surfaces the Riemann tensor, $R_{\mu\nu\sigma\rho}$, possesses only one independent component.

The line element of a two dimensional surface with coordinates $x^\mu = (\theta, \phi)$ is given by

$$ds^2 = d\theta^2 + \sec^2\theta\, d\phi^2 \ .$$

Write down the metric tensor and its inverse. Show that $\Gamma^\theta_{\phi\phi} = -\sec^2\theta\tan\theta$ and $\Gamma^\phi_{\theta\phi} = \tan\theta$ and compute the remaining components of the Christoffel symbol, $\Gamma^\mu_{\nu\sigma}$.

Show that

$$R^\theta{}_{\phi\theta\phi} = -(2\sec^2\theta - 1)\sec^2\theta \ .$$

Hence deduce the value of $R^\phi{}_{\theta\phi\theta}$. With these expressions, compute $R_{\mu\nu}$ and R and show that the Einstein tensor vanishes.

6. A particle of mass m moves along a geodesic of a spacetime with metric $g_{\mu\nu}$ none of whose components depend on a particular coordinate x^α. Show that the corresponding component of momentum, namely p_α, is constant on each geodesic.

If the particle moves freely in the equatorial plane, $\theta = \pi/2$, of the Schwarzschild spacetime, whose line element is

$$(ds)^2 = \left(1 - \frac{2M}{r}\right) c^2 (dt)^2 - \left(1 - \frac{2M}{r}\right)^{-1} (dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2\theta \, (d\phi)^2$$

where M is constant, deduce that p_0 and p_ϕ are constant.

If $p_0 c = m\tilde{E}$, $p_\theta = 0$ and $p_\phi = -\, m\tilde{L}$, show that $dr/d\tau$ satisfies an equation of the form

$$c^2 \left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - c^4 W(r)$$

and deduce $W(r)$, where τ is the proper time.

For $\tilde{L}^2 > 12M^2c^2$, sketch all possible forms of $W(r)$ in the region $r > 2M$, and discuss the nature of all types of particle trajectories. For what range of $\tilde{L}$ are hyperbolic orbits possible?

7. Consider a photon moving in the equatorial plane of the Schwarzschild spacetime. The trajectory is governed by

$$\left(\frac{dr}{d\lambda}\right)^2 = \tilde{E}^2 - \frac{\tilde{L}^2}{r^2}\left(1 - \frac{2M}{r}\right) \quad \text{and} \quad \left(\frac{d\phi}{d\lambda}\right) = \frac{\tilde{L}}{r^2}$$

where M is the mass of the gravitational source, $c = 1$ and λ parametrises the worldline of the photon. Compute $d\phi/dr$ in terms of r, M and $b = \tilde{L}/\tilde{E}$. What is the physical interpretation of b?

For an incoming photon with $\tilde{L} > 0$ and $d\phi/dr < 0$, show that

$$\frac{d\phi}{du} = \left(\frac{1}{b^2} - u^2(1 - 2Mu)\right)^{-1/2}$$

where $r = 1/u$. Solve the equation in the case when $M = 0$ and describe the geometry of the trajectory.

For $M \neq 0$, show that

$$\frac{d\phi}{dy} = (1 + 2My)\left(\frac{1}{b^2} - y^2\right)^{-1/2}$$

where $y = u(1 - Mu)$ and terms of order y^3 and higher are neglected in this substitution. If initially $\phi = \phi_0$ at $r = \infty$, show that

$$\phi = \phi_0 + \sin^{-1}(by) + \frac{2M}{b} - 2M\left(\frac{1}{b^2} - y^2\right)^{1/2} .$$

What is the angle of deflection at the point of closest approach?