2MA65 QUANTUM MECHANICS JANUARY 1999

In this paper bold-face quantities like $\mathbf{r}$ represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. A particle of mass m is confined to the region $0 \leq x \leq L$ of the x-axis. Find the normalised eigenfunctions of the Hamiltonian, and show that the energy eigenvalues are E_n where

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad n = 1, 2, 3 \ldots$$

At a certain time the particle is in a state described by the normalised wave-function

$$\psi(x) = Ax \qquad 0 \leq x \leq \frac{L}{2}$$

$$\psi(x) = A(L - x) \qquad \frac{L}{2} \leq x \leq L$$

$$\psi(x) = 0 \qquad x < 0 \quad \text{and} \quad x > L,$$

where A is real.

(i) Determine the normalisation constant A.

(ii) Calculate the probability that a measurement of the energy will give the result E_1.

2. A particle of mass m and energy $E < 0$ moves on the x-axis subject to a potential V given by

$$V = 0 \qquad |x| > a$$

$$V = -V_0 \qquad |x| \leq a$$

where V_0 and a are positive constants. Suppose that $E > -V_0$. Define

$$q^2 = -\frac{2mE}{\hbar^2} \qquad \text{and} \qquad k^2 = \frac{2m(E + V_0)}{\hbar^2}.$$

(i) Write down the energy eigenfunction equation in the regions $|x| \leq a$ and $|x| > a$. Hence show that the energy eigenfunctions are either odd or even functions of x.

(ii) Show that for an odd solution, k must satisfy

$$k \cot(ka) = -\sqrt{\alpha^2 - k^2},$$

where

$$\alpha^2 = \frac{2mV_0}{\hbar^2}.$$

3. The Hamiltonian for a particle of mass m moving on the x-axis in a harmonic oscillator potential is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

where

$$p = -i\hbar\frac{d}{dx}$$

and ω is a positive constant.

(i) Show that if we define

$$a = \frac{1}{\sqrt{2}}\left(\frac{1}{\hbar\alpha}p - i\alpha x\right) \qquad \text{and} \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\hbar\alpha}p + i\alpha x\right),$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$, then it follows from the basic commutator $[x, p] = i\hbar$ that $[a, a^\dagger] = 1$.

(ii) Show by induction that $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$.

(iii) The normalised eigenfunctions of the Hamiltonian are given by

$$\psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_0,$$

where $a\psi_0 = 0$. Show that

$$a\psi_n = \sqrt{n}\psi_{n-1} \qquad \text{and} \qquad a^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}.$$

(iv) Write

$$(a + a^\dagger)^2\psi_n$$

in terms of ψ_{n-2}, ψ_n and ψ_{n+2}. Hence show that

$$< \psi_n|p^4|\psi_n >= \frac{3}{4}\hbar^2 m^2\omega^2(2n^2 + 2n + 1).$$

[You may find the following identity useful:

$$[A, BC] = B[A, C] + [A, B]C$$

for operators A, B and C.]

4. The angular momentum operators L_1, L_2 and L_3 satisfy the commutation relations

$$[L_1, L_2] = i\hbar L_3 \qquad \text{and cyclic permutations,}$$

which imply

$$[\mathbf{L}^2, L_1] = [\mathbf{L}^2, L_2] = [\mathbf{L}^2, L_3] = 0$$

(where $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$).

Suppose that $|l, m>$ are the normalised eigenstates such that

$$L_3|l, m> = \hbar m|l, m>, \qquad \mathbf{L}^2|l, m> = \hbar^2 l(l+1)|l, m> .$$

($\mathbf{L}^2$ and L_3 form a complete commuting set of observables, so that these properties define $|l, m>$ uniquely.)

(i) Defining $L_+ = L_1 + iL_2$ and $L_- = L_1 - iL_2$, show that

$$[L_3, L_+] = \hbar L_+, \quad [L_3, L_-] = -\hbar L_-.$$

Hence, using also the commutation relations for $\mathbf{L}^2$ above, deduce that

$$L_+|l, m> = N_{l,m}|l, m+1>$$

and

$$L_-|l, m> = M_{l,m}|l, m-1>,$$

where $N_{l,m}$ and $M_{l,m}$ are constants.

(ii) A particle is in the normalised angular momentum eigenstate

$$|\psi> = z|1, -1> + z^*|1, 1> + c|1, 0>$$

where $z = a + ib$, and a , b and c are real. Show that the expectation value $< L_3 >$ of L_3 in this state is zero. By writing L_1 and L_2 in terms of L_+ and L_-, compute $< L_1 >$ and $< L_2 >$ for this state in terms of a, b and c.

[You may assume that in (i), $N_{l,m}$ and $M_{l,m}$ are given by

$$N_{l,m} = \hbar\sqrt{l(l+1) - m^2 - m}, \qquad M_{l,m} = \hbar\sqrt{l(l+1) - m^2 + m}.]$$

5. The Hamiltonian for a stationary electron of mass m and charge e in a constant magnetic field B along the z-axis is given by $H = \hbar\omega\sigma_3$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\omega = \frac{eB}{2m}$.

(i) By solving Schrödinger's equation, show that at time t the state of the electron is given by

$$\psi(t) = \begin{pmatrix} c_1 e^{-i\omega t} \\ \\ c_2 e^{i\omega t} \end{pmatrix},$$

where c_1, c_2 are constants.

(ii) A certain observable is represented by $A = \alpha \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, where α is a real constant. Compute the eigenvalues and normalised eigenvectors of A. What are the possible results of a measurement of A?

(iii) At time $t = 0$ the observable A is measured, giving a result 5α. The system is then left undisturbed until time t. What is the expectation value of A at time t?

6. The Hamiltonian for a particle of mass m moving in a three-dimensional harmonic oscillator potential is

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2,$$

where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

(i) Given that the ground state wave function is $\psi(r) = Ae^{-\frac{1}{2}\beta^2 r^2}$, where A is real, determine β and the ground state energy E_0.

(ii) Calculate the normalisation constant A.

(iii) The potential is perturbed by the addition of a term $\lambda V(r)$, where $V(r) = r^4$ and λ is a small parameter. Show that to first order in λ, the ground state energy is now given by

$$E = E_0 + \lambda \frac{15}{4} \frac{\hbar^2}{m^2\omega^2}.$$

[Standard results in perturbation theory may be assumed without proof. Moreover, defining $I_n = \int_0^\infty r^n e^{-\beta^2 r^2}\, dr$, you may assume that

$$I_0 = \frac{\sqrt{\pi}}{2\beta} \qquad \text{and}$$

$$I_n = \frac{n-1}{2\beta^2} I_{n-2} \qquad (n \geq 2).$$

You may also assume that the radial part of the Laplacian in spherical polars is

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.]$$

7. A particle of mass m moves on the x-axis subject to a potential

$$V(x) = \lambda |x^3|,$$

where λ is a positive constant.

Consider a normalised wave function of the form

$$\psi(x) = A(a^2 - x^2) \qquad (|x| \leq a)$$
$$\psi(x) = 0 \qquad \text{otherwise},$$

where A is real.

(i) Compute the normalisation constant A.

(ii) Show that with this wave function, the expectation value of the Hamiltonian is given by

$$< H >= \frac{5}{4}\frac{\hbar^2}{ma^2} + \frac{5\lambda a^3}{64}.$$

(iii) Hence use the variational principle to show that an estimate for the ground state energy is given by

$$E_0 \approx \frac{25}{16}\left(\frac{\hbar^6 \lambda^2}{27m^3}\right)^{\frac{1}{5}}.$$

Is the true ground state energy less than, or greater than, this value?