1. Show that, in the game Breton poker, described at the end of this question, the probability of A drawing two red cards is $\frac{1}{5}$.

Draw the game tree for Breton poker. Where random moves are made, indicate the probabilities. Write down the pure strategies for each player.

Note that if A draws cards of the same colour she can force a win: eliminate those strategies in which she then folds. Why should you also eliminate those of B's strategies in which if she draws cards of the same colour she raises? Having eliminated these strategies, construct the payoff matrix for the reduced game.

Rules for Breton Poker. The game is played by two players, A and B, with a set of six cards, three of which are red and three black. The cards are put in a hat and each player puts £5 in the kitty. A draws two cards from the hat without showing them to B, and then either **folds**, in which case B wins the kitty, or **raises** by adding £10 to the kitty. If A raises, then B draws two cards from the hat, looks at them, and then either folds, so that A wins the kitty, or raises by adding £10 to the kitty. If this stage is reached, and if each player has cards of different colours, then B wins the kitty, but if one or both players have cards of the same colour, then A wins.

2. The game players Archie and Bernard have VNM utility functions U_A and U_B, respectively, where the values for the sure prospects s_i, $i = 1, 2, ..., 6$, are given in the table:

	s_1	s_2	s_3	s_4	s_5	s_6
U_A	1	4	-2	-5	2	4
U_B	3	1	5	7	1	-3

Let $s(p)$ denote the risky prospect $[ps_2, (1-p)s_4]$.

Plot the function $U_A(s(p))$ for $0 \leq p \leq 1$, marking the values of p for the risky prospects, with respect to s_2 and s_4, which are equivalent to each of the sure prospects s_i, $i = 1, 2, ..., 6$. Why should s_6 not appear on a plot of $U_B(s(p))$?

Let the payoff pairs $(U_A(s_i), U_B(s_i))$ be denoted by t_i, for $i = 1, 2, ...6$.

The bimatrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are given by

$$\mathbf{M}_1 = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} t_1 & t_3 \\ t_5 & t_6 \end{pmatrix}.$$

Archie and Bernard play two games: Game 1 has payoff matrix $\mathbf{M}_1$; Game 2 is a cooperative game with payoff matrix $\mathbf{M}_2$.

Draw the attainment sets for Games 1 and 2.

Deduce that Game 1 is completely antagonistic. Transform Game 1 to an explicitly zero sum game and solve it.

Find the Pareto Optimal Set for Game 2.

3. Two croquet clubs, the Anfield Avengers and the Goodison Gorgons, each has three teams at its disposal. Every week throughout the summer, each club captain has to select a team to take part in a Saturday afternoon match, without knowing which team the other club captain is likely to select. The probablities of a win by the Avengers, corresponding to the various possible pairings of opposing teams, known from past experience, are given by the following form matrix, in which the rows are labelled by teams from the Avengers club and the columns by teams from the Gorgons club:

$$\begin{pmatrix} 0.8 & 0.2 & 0.4 \\ 0.4 & 0.5 & 0.6 \\ 0.1 & 0.7 & 0.3 \end{pmatrix}.$$

Regarding this as the payoff matrix for the row player of a two-player game, show that the game is equivalent to a zero sum game.

Find the frequencies with which the club captains should select the various teams in order to obtain the greatest number of victories on average. Which is the more successful club?

4. (i) Denote by A and B the participants in a two-player noncooperative game. Let the payoffs to the players be $V_A(\mathbf{p}, \mathbf{q})$ and $V_B(\mathbf{p}, \mathbf{q})$, respectively, when A plays mixed strategy $\mathbf{p}$ against B's mixed strategy $\mathbf{q}$. Express in algebraic form the statement that the pair $(\mathbf{p}^*, \mathbf{q}^*)$ is an equilibrium point of the game.

Assuming the existence of at least one equilibrium point, prove, **in the case of a zero sum game**, that all equilibrium points have the same game value.

(ii) Find the equilibrium points of the following noncooperative games and determine which are Pareto optimal.

$$\text{(a)} \quad \begin{pmatrix} (2,1) & (-1,-1) \\ (-1,-1) & (1,2) \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} (0,-3) & (1,0) \\ (1,1) & (0,0) \end{pmatrix}.$$

Which, if any, of these games has a solution in the strict sense?

5. A 2-player cooperative game has bimatrix:

$$\begin{pmatrix} (5,1) & (7,4) & (1,10) \\ (1,1) & (9,-2) & (5,1) \end{pmatrix}.$$

Draw the attainment set for this game and mark the Pareto optimal set. Given that the maximin values for this game are 3 for the row player and 1 for the column player, indicate the negotiation set. Calculate the maximin bargaining payoffs.

The column player decides to discard his second strategy because he feels that it overly rewards the row player. The latter responds by proposing that the outcome to the game should be determined by threat bargaining. Which player loses most from this turn of events?

6. Explain the superadditivity property of the characteristic function of an n-person cooperative game. What is a **dummy** in such a game?

Five boys, Alec, Basil, Cecil, Dennis, and Eric, meet behind the scout hut to consider the exchange of champion conkers.

Alec and Basil each have such a conker, which they value at 4 gobstoppers and 6 gobstoppers, respectively. Cecil, Dennis, and Eric do not possess a conker, but each would like to acquire one (and only one) and would value one at 5 gobstoppers, 3 gobstoppers, and 7 gobstoppers, respectively.

Regard the meeting of the boys as a 5-person cooperative game with transferable utilities, and for which each boy's utility has zero contribution from an initial holding of gobstoppers.

Justify the statement that one boy is a dummy in this game.

Calculate the values of the characteristic function for the 4-person game without the dummy.

Write down the conditions for an imputation to be in the core.

Show that, in the core, Basil gets six gobstoppers and Cecil gets none.

Given that gobstoppers are indivisible, what does the core predict for the other players?

7. Abdul and Benjamin are traders in sheep and goats at the Shalom cattle market. On one particular day, Abdul comes to market with 50 sheep and 10 goats, and Benjamin comes with 30 sheep and 70 goats.

The traders' preferences for sheep and goats are represented by the utility functions:

$$U_A(s, g) = (s + 15)(2g + 10), \; U_B(s', g') = (s' + 10)(g' + 50),$$

where s, s' are the holdings in sheep, and g, g' are the holdings in goats.

Draw the Edgeworth box and sketch very roughly a few indifference curves for each trader.

Show that the Pareto optimal set for trading without money is the line $7g - 9s = 100$. Determine the end points of the contract curve.