PAPER CODE NO.
2MA2C

THE UNIVERSITY
of LIVERPOOL

JANUARY 1998 EXAMINATIONS

Degree of Bachelor of Engineering: Year 2
Degree of Master of Engineering: Year 2

Mathematics for Civil Engineers

Time allowed: 3 hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for *six* complete answers. Only the best *six* answers will be taken into account.

A statistical table is attached to the back of this paper.

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1. (a) Find the (unique) solution to

$$\begin{aligned} 3x - 2y + z &= 6, \\ x + 10y - z &= 2, \\ -3x - 2y + z &= 0, \end{aligned}$$

by using elementary row operations.

(b) Find the general solution of

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 1, \\ 2x_1 + x_2 + x_3 \phantom{{}+2x_4} &= 2, \\ -x_1 + 2x_2 + x_3 + 2x_4 &= 3, \end{aligned}$$

by using elementary row operations; express your answer in vector form and interpret it geometrically.

2. (a) Find the (unique) solution to

$$\begin{pmatrix} 2 & 5 & -1 \\ 1 & 4 & -3 \\ 2 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 15 \end{pmatrix}$$

by first finding the *adjoint* and *determinant* of the square matrix, and then the inverse, $\mathbf{A}^{-1}$.

(b) Find the solution once more, this time using Cramer's rule.

(c) Check that the solution from Part (a) is the same as the solution from Part (b).

3. A uniform string is clamped at its ends $x = 0$ and $x = l$. Its displacement, $u(x,t)$, at time t is governed by

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

(a) Find the most general solution of the form

$$u(x,t) = X(x)T(t)$$

which satisfies the given boundary conditions.

(b) Given the initial conditions

$$u(x,0) = J \sin\left(\frac{\pi x}{l}\right), \qquad \frac{\partial u}{\partial t}(x,0) = K \sin\left(\frac{\pi x}{l}\right),$$

find the particular solution which satisfies these.

4. A mass is attached to a damped spring and is externally forced. Motion is governed by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 0.02\frac{\mathrm{d}y}{\mathrm{d}t} + 40y = f(t),$$

where

$$f(t) = \begin{cases} 0, & \text{in } -T \le t < 0; \\ 1, & \text{in } 0 \le t < T \end{cases}$$

and

$$f(t + 2T) = f(t) \qquad \forall t.$$

(a) Find the Fourier series of the forcing function, $f(t)$.

(b) By using your answer to Part (a) find the motion of the mass (you may neglect the transient part of the solution).

5. To find maxima or minima of an integral

$$I = \int_{x_1}^{x_2} f(x, y, y_x)\,\mathrm{d}x$$

one usually has to solve the Euler-Lagrange equation,

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_x}\right) = 0, \qquad x_1 \le x \le x_2.$$

(a) Show why, if f does not explicitly depend upon x, one only needs to solve the simpler equation

$$y_x \frac{\partial f}{\partial y_x} - f = k,$$

where k is a constant.

(b) Show that the curve $y = y(x)$ in the (x, y)-plane for which the integral

$$\int_0^2 \frac{(1 + y_x^2)^{1/2}}{1 + y}\,\mathrm{d}x$$

is minimised is the arc of a circle.

6. (a) Show that the point $(1, 2)$ is a saddle of the function

$$f(x, y) = \tfrac{1}{2}xy^2 - x^2y - \tfrac{1}{2}y^2 + xy + 2x^2 - 4x.$$

(b) Find and classify the other three stationary points of $f(x, y)$.

7. (a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy} and f_{yyy} of the function

$$f(x, y) = y \ln(x + y).$$

(b) Use your results from Part (a) to find the Taylor Series at (0,1) for f up to and including terms cubic in the increments δx and δy.

(c) Use the approximation to the Taylor Series found in Part (b) to obtain linear, quadratic and cubic approximations for $f(0.2, 1.1)$.

8. (a) (i) Find k so that

$$f(x) = \begin{cases} ke^{-\lambda x}, \lambda > 0, & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is a permissible probability density function (p.d.f.).

(ii) Sketch the p.d.f. and the cumulative density function (c.d.f).

(iii) Find the probability that x lies in the range $(1, 3)$, to 3 s.f.

(b) Experience has shown that on average four pairs of men's shoes and three pairs of ladies' shoes are left for repair at a cobbler's shop during a particular hour of the day.

(i) Suppose that these events may be satisfactorily modelled by a Poisson distribution,

$$P(x) = \mathrm{e}^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots.$$

Determine the probability density distribution, for shoes left, Y. State any assumptions you make.

(ii) What is the probability that not more than a total of four pairs of shoes will be left during a particular hour on any day? State your answer to 3 s.f.

9. An engineering firm produces widgets, the nominal length of which is 5.10 mm and the tolerance of which is ± 0.04 mm; any widgets outside the interval (5.06, 5.14)mm are rejected.

(a) The measured lengths (to the nearest 0.01 mm) of 50 widgets are given in Table 1. How many widgets will be rejected?

(b) Group the data and draw the corresponding histogram.

(c) Calculate the mean and standard deviation of the grouped data to 3 s.f.

(d) Assume the population is normally distributed with mean and standard deviation equal to the sample values you have calculated. Hence show that approximately 37% of widgets will be rejected.

5.13	5.10	5.15	5.13	5.15
5.11	5.09	5.15	5.10	5.12
5.11	5.17	5.14	5.12	5.13
5.13	5.13	5.13	5.15	5.13
5.12	5.13	5.12	5.11	5.12
5.14	5.10	5.13	5.13	5.14
5.16	5.10	5.12	5.11	5.13
5.14	5.16	5.11	5.14	5.13
5.14	5.12	5.14	5.11	5.16
5.12	5.13	5.15	5.15	5.12

Table 1: Measured length of 50 widgets, in mm, to the nearest 0.01 mm.

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