2MA1E

Instructions to candidates

Answer all of section A and THREE questions from section B. The total of marks available in section A is 55.

SECTION A

1. Sketch the graph of the function

$$f(x) = \cos^2 x.$$

State its domain and range.

[4 marks]

2. Obtain the Maclaurin series expansion of the function

$$f(x) = \sin x \;\; \cosh x,$$

up to and including terms in x^3.

[5 marks]

3. State, with reasons, whether the following functions are odd, even or neither:

$$(a)\ e^{\cos x}, \qquad (b)\ x^{123} + x^{50}, \qquad (c)\ \sin^4 x.$$

[6 marks]

4. Calculate the integral

$$\int_0^{\frac{3}{2}} (x+2)^3 \;\; dx,$$

evaluating the result to three decimal places.

[5 marks]

5. Which of the following limits exist? Evaluate those that do.

$$(a)\ \lim_{x \to +\infty} \frac{x^2}{e^x}, \qquad (b)\ \lim_{x \to +\infty} \sin x \;\; \cos x, \qquad (c)\ \lim_{x \to 0} \frac{x}{1 - e^{-x}}.$$

[6 marks]

6. Differentiate the following functions:

$$(a)\ e^{\cos x}, \qquad (b)\sqrt{\ln x}, \qquad (c)\ \frac{\sin x}{\sinh^2 x}.$$

[6 marks]

7. Find the equation of the tangent to the curve

$$xy + \log y = 0$$

at the point $(x, y) = (0, 1)$.

[5 marks]

8. Let $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$. Find $\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$ and $\mathbf{a} \cdot \mathbf{b}$. Calculate (in radians, to 3 decimal places) the angle between $\mathbf{a}$ and $\mathbf{b}$.

Give examples of a vector parallel to $\mathbf{a}$ and a vector orthogonal to $\mathbf{b}$.

(*Hint:* if two vectors are orthogonal their scalar product vanishes...)

[8 marks]

9. (i) Let $z_1 = 3 + i$ and $z_2 = -2 - 5i$. Express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $a + ib$ where a and b are real.

(ii) Evaluate $(-\frac{1}{\sqrt{6}} + \frac{i}{\sqrt{6}})^7$. [*Hint:* use De Moivre's theorem.]

[5 marks]

10. Find the polar form of all complex numbers z such that

$$z^3 = \frac{1}{4}\,(-1 + i).$$

Indicate their positions in an Argand diagram.

[5 marks]

SECTION B

13. Let $f(x) = e^x - \sinh x$.

Calculate the integral

$$\int_0^2 f(x)\ dx,$$

evaluating the result to three decimal places.

Obtain the Maclaurin series expansion of $f(x)$ up to and including terms in x^3 in the form

$$a + bx + cx^2 + dx^3,$$

where a, b, c, d are real constants that you should determine.

Then evaluate

$$\int_0^2 \left(a + bx + cx^2 + dx^3\right)\ dx$$

to three decimal places.

Estimate the difference between the two integrals. Do they agree within 1%?

[15 marks]

14. Determine all the (real) values of x such that the series

$$\sum_{n=1}^{\infty} \frac{(x^2-5)^n}{n}$$

converges.

Explain your reasoning.

(*Hint:* It may be helpful to you to sketch the graph of $f(x) = x^2 - 5$.)

[15 marks]

15. By sketching the graphs of $y = \cosh x$ and $y = -x^2 + 3$, demonstrate that the equation

$$f(x) = \cosh x + x^2 - 3 = 0$$

has two solutions (for real x).

Use the Newton-Raphson formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

to find the approximate value of the solution which occurs for a negative value of x. You are asked to choose an appropriate approximation to such a solution, x_0, to use the Newton-Raphson formula to find x_1, and to use x_1 in turn to find another approximation x_2.

Test whether x_2 is in fact a better approximation to the exact result than x_0.

Without performing any additional calculation, what is your best guess for the value of the other solution?

[15 marks]

16. Determine the values of two (real) constants a, b such that the function $f(x)$, defined as

$$\begin{cases} a \ \sin x + b \ \cos x & \text{for } x < \frac{\pi}{2} \\ \ln\left(\frac{\pi}{2} - x - 1\right) & \text{for } x \geq \frac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$ and $f'(\frac{\pi}{2})$ exists.

Hence sketch the graph of $y = f(x)$, indicating clearly the positions of asymptotes, stationary points, points of inflection and zeros. Does $f(x)$ have any vertical asymptote? What is the domain of $f(x)$?

[15 marks]

17. Use De Moivre's theorem to determine the (real) constants a, b such that

$$\sin 4\theta \ \cos 4\theta = a(\cos^7\theta \ \sin\theta - \sin^7\theta \ \cos\theta) + b(\cos^5\theta \ \sin^3\theta - \cos^3\theta \ \sin^3\theta).$$

Check your result for $\theta = \frac{\pi}{4}$.

[15 marks]