2MA1E

Instructions to candidates

Answer all of section A and THREE questions from section B. The total of marks available in section A is 55.

SECTION A

1. Sketch the graph of the function

$$f(x) = \sqrt{2 - x}.$$

State its domain.

[3 marks]

2. Obtain the Maclaurin series expansion of the function

$$f(x) = x \;\ln(1 + x),$$

up to and including terms in x^3.

[4 marks]

3. State, with reasons, whether the following functions are odd, even or neither:

$$(a)\;\ln(1 + x^2), \qquad (b)\;\sin x \;\sinh x, \qquad (c)\; x^3 + \cos x.$$

[6 marks]

4. Calculate the integral

$$\int_0^{\pi/2} (\sin x - 3x)\; dx,$$

evaluating the result to three decimal places.

[4 marks]

5. Which of the following limits exist? Evaluate those that do.

$$(a)\;\lim_{x \to 1} \frac{x^2 + 4x - 2}{x - 1}, \qquad (b)\;\lim_{x \to -2} \frac{x^2 + 7x + 10}{x + 2}, \qquad (c)\;\lim_{x \to 0} \frac{\sin x}{1 - e^x}.$$

[6 marks]

6. Differentiate the following functions:

$$(a)\ e^{\sin x}, \qquad (b)\ \frac{\ln x}{\sin x}, \qquad (c)\ \sqrt{\cosh x}.$$

[5 marks]

7. Prove that the function

$$f(x) = \ln x - x^3$$

has one (and only one) stationary point.

Determine its nature.

[4 marks]

8. Find the equation of the tangent to the curve

$$2 + x \ln y - y = 0$$

at the point $(x, y) = (0, 2)$.

[5 marks]

9. Find the general solution (for real θ) of the equation

$$\sin\theta + \frac{1}{\sqrt{3}}\cos\theta = 0.$$

[3 marks]

10. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{a} + \mathbf{j}$. Find $\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$ and $\mathbf{a} \cdot \mathbf{b}$. Calculate (in radians, to 3 decimal places) the angle between $\mathbf{a}$ and $\mathbf{b}$.

[6 marks]

11. (i) Let $z_1 = 2 - 3i$ and $z_2 = 1 + i$. Express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $a + ib$ where a and b are real.

(ii) Evaluate $(1 + i)^{10}$. [*Hint:* use De Moivre's theorem.]

[4 marks]

12. Find the polar form of all complex numbers z such that

$$z^3 = 1 - i.$$

Indicate their positions in an Argand diagram.

[5 marks]

SECTION B

13. Let $f(x) = 3\ e^x$.

Calculate the integral

$$\int_0^{1/2} f(x)\ dx,$$

evaluating the result to three decimal places.

Obtain the Maclaurin series expansion of $f(x)$ up to and including terms in x^3 in the form

$$a + bx + cx^2 + dx^3,$$

where a, b, c, d are real constants that you should determine.

Then evaluate

$$\int_0^{1/2} \left(a + bx + cx^2 + dx^3\right)\ dx$$

to three decimal places.

Estimate the difference between the two integrals. Do they agree within 1%?

[15 marks]

14. Determine all the (real) values of x such that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1-x}{1+x}\right)^n$$

converges.

Explain your reasoning. [*Hint:* you may want to sketch the graph of $f(x) = \frac{1-x}{1+x}$].

[15 marks]

15. By sketching the graphs of $y = \frac{1}{2}\cosh x$ and $y = \cos x$, demonstrate that the equation

$$f(x) = \frac{1}{2}\cosh x - \cos x = 0$$

has two solutions (for real x).

Use the Newton-Raphson formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

to find the approximate value of the solution which occurs for a positive value of x. You are asked to choose an appropriate approximation to such a solution, x_0, to use the Newton-Raphson formula to find x_1, and to use x_1 in turn to find another approximation x_2.

Test whether x_2 is in fact a better approximation to the exact result than x_0.

Without performing any additional calculation, what is your best guess for the value of the other solution?

[15 marks]

16. Let $f(x)$ be defined as

$$\begin{cases} \frac{x}{x-1} & \text{for } x \geq 0 \\ -\frac{x}{x-1} & \text{for } x < 0 \end{cases}$$

Sketch the graph of $y = f(x)$, indicating clearly the positions of asymptotes and zeros.

Likewise, sketch the graph of $y = f'(x)$.

Show that

$$\lim_{x \to 0} f'(x)$$

does not exist, and explain what features of the graphs are a consequence of this result.

[15 marks]

17. Use De Moivre's theorem to determine the (real) constant a such that

$$\cos 3\theta = \cos^3\theta - a\cos\theta\ \sin^2\theta.$$

Show also that

$$\tan 3\theta = \tan\theta\left(\frac{a\cos^2\theta - \sin^2\theta}{\cos^2\theta - a\sin^2\theta}\right).$$

Check both results for $\theta = \frac{\pi}{4}$.

[15 marks]