2MA1E

Instructions to candidates

Answer all of section A and THREE questions from section B. The total of marks available in section A is 55.

SECTION A

1. Sketch the graph of the function

$$f(x) = |\sin x|.$$

State its domain.

[3 marks]

2. Obtain the first three nonzero terms in the Maclaurin series expansion of the function

$$f(x) = \sqrt{x^2 + x + 1}.$$

[4 marks]

3. State, with reasons, whether the following functions are odd, even or neither:

$$(a)\ \frac{\sin x}{\sqrt{3+x^2}}, \qquad (b)\ e^x + x^4, \qquad (c)\ \tanh(x^3).$$

[6 marks]

4. Calculate the integral

$$\int_1^4 (e^x - \frac{3}{x^2})\ dx,$$

evaluating the result to three decimal places.

[4 marks]

5. Which of the following limits exist? Evaluate those that do.

$$(a)\ \lim_{x\to 1} \frac{x^5 - 2x + 1}{6x^4 - 3x^2 - 3}, \qquad (b)\ \lim_{x\to 0} \frac{\tan x}{\tanh x}, \qquad (c)\ \lim_{x\to +\infty} \frac{\sin x}{x}.$$

[6 marks]

6. Differentiate the following functions:

$$(a)\ \cosh x\ \cos(x^2), \qquad (b)\ \ln(\sqrt{x}), \qquad (c)\ \frac{x^2}{\sin^2 x}.$$

[6 marks]

7. Prove that the function

$$f(x) = \frac{1}{(x+1)} - \frac{1}{(x-1)}$$

has one (and only one) stationary point.

Determine its nature.

[4 marks]

8. Find the equation of the tangent to the curve

$$\sin(x^2) + \ln y - \frac{\pi}{2}x = 0$$

at the point $(x, y) = (0, 1)$.

[5 marks]

9. Find the absolute minimum of the function

$$f(x) = 3x^3 - 2x^2$$

in the domain $-1 \leq x \leq 1$.

[3 marks]

10. Let $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$ and $\mathbf{a} \cdot \mathbf{b}$. Calculate (in radians, to 3 decimal places) the angle between $\mathbf{a}$ and $\mathbf{b}$.

[6 marks]

11. Define $\cos\theta$ and $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

From the definitions, prove that

$$2\sin\theta\ \cos\theta = \sin(2\theta).$$

[3 marks]

12. Find the polar form of all complex numbers z such that

$$z^3 = 4\sqrt{3} - 4i.$$

Indicate their positions in an Argand diagram.

[5 marks]

SECTION B

13. The function $y = f(x)$ is defined parametrically through the equations

$$\begin{cases} x(t) = 2t + 3t^2 \\ y(t) = t^2 + 2t^3 \end{cases}$$

Prove that

$$y = (\frac{dy}{dx})^2 + 2(\frac{dy}{dx})^3.$$

Find the equations of the lines which are tangent to the graph of $y = f(x)$ at the point $(x, y) = (5, 3)$ and at the point $(x, y) = (16, 20)$ respectively.

Hence find the coordinates of the point of intersection of the two tangent lines and calculate the angle (in radians, to 3 decimal places) between the lines.

[15 marks]

14. Determine all the (real) values of x such that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-2)^{2n}}{2n}$$

converges. Explain your reasoning.

Hence determine all the (real) values of x such that the above series is absolutely convergent.

[15 marks]

15. By sketching the graphs of $y = x^3$ and $y = \tan^{-1} x$, demonstrate that the equation

$$f(x) = x^3 - \tan^{-1} x = 0$$

has three solutions (for real x).

Use the Newton-Raphson formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

to find the approximate value of the solution which occurs for a positive value of x. You are asked to choose an appropriate approximation to such a solution, x_0, to use the Newton-Raphson formula to find x_1, and to use x_1 in turn to find another approximation x_2. [*Hint:* you are advised not to use for the initial guess x_0 a value smaller than 1. Remember that $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$].

Test whether x_2 is in fact a better approximation to the exact result than x_0.

Without performing any additional calculation, state your best guess for the values of the other two solutions of $f(x) = 0$.

[15 marks]

16. Determine the values of two (real) constants a, b such that the function $f(x)$, defined as

$$\begin{cases} ax^3 + bx^2 & \text{for } x \leq 2 \\ \frac{x}{(1-x)} & \text{for } x > 2 \end{cases}$$

is continuous at $x = 2$ and $f^{'}(2)$ exists.

Hence sketch the graph of $y = f(x)$, indicating clearly the positions of asymptotes, stationary points, points of inflection and zeros. Does $f(x)$ have any vertical asymptote?

[15 marks]

17. Use De Moivre's theorem to determine the (real) constants a, b, c such that

$$\cos 3\theta \sin 4\theta = \cos^2\theta \sin\theta (a\cos^4\theta + b\cos^2\theta\sin^2\theta + c\sin^4\theta)$$

Hence check the identity for $\theta = \frac{\pi}{3}$.

[15 marks]