2MA1E

Instructions to candidates

Answer all of section A and THREE questions from section B. The total of marks available in section A is 55.

SECTION A

1. Sketch the graph of the function

$$f(x) = \ln(|x|).$$

State its domain.

[3 marks]

2. Obtain the Maclaurin series expansion of the function

$$f(x) = (3 - 2x)^{-3},$$

up to and including terms in x^3.

[4 marks]

3. State, with reasons, whether the following functions are odd, even or neither:

$$(a)\ \sqrt{x^2+1}, \qquad (b)\ \frac{1-x^2}{1+x^4}, \qquad (c)\ \sinh(x^3).$$

[6 marks]

4. Calculate the integral

$$\int_1^3 (3x^2 - \frac{1}{x})\,dx,$$

evaluating the result to three decimal places.

[4 marks]

5. Which of the following limits exist? Evaluate those that do.

$$(a)\ \lim_{x\to 1} \frac{x^3 - 2x^2 - x + 2}{x^3 - 7x + 6}, \qquad (b)\ \lim_{x\to 3} \frac{x^2 - x - 6}{x - 3}, \qquad (c)\ \lim_{x\to +\infty} \frac{\ln(e^{x^2})}{x^2+3}.$$

[6 marks]

6. Differentiate the following functions:

$$(a)\ \frac{\sin(x^3)}{x^2}, \qquad (b)\ \frac{\sin^3 x}{x^3}, \qquad (c)\ e^{\sqrt{x}}.$$

[5 marks]

7. Prove that the function

$$f(x) = x^3 + 3x^2 + 3x + 7$$

has one (and only one) stationary point.

Determine its nature.

[4 marks]

8. Find the equation of the tangent to the curve

$$x \sin y + \sin x = 0$$

at the point $(x, y) = (\pi, 0)$.

[5 marks]

9. Define $\cosh x$ and $\sinh x$ in terms of e^x and e^{-x}.

From the definitions, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

[3 marks]

10. Use the ratio test to prove that the series

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \cdots + \frac{2n-1}{2^n} + \cdots$$

converges.

[5 marks]

11. Solve the equation

$$z^2 + 2z + 2 = 0,$$

where z is a complex number, using the quadratic formula.

Check your result by substitution.

[4 marks]

12. Find the polar form of all complex numbers z such that

$$z^4 = -8 + 8i\sqrt{3}.$$

Indicate their positions in an Argand diagram.

[6 marks]

SECTION B

13. (i) Find four solutions of the equation

$$z^4 + 8 = 0,$$

where z is a complex number.

Use your results to factor $z^4 + 8 = 0$ into two quadratic factors with real coefficients.

[10 marks]

(ii) Show that if w and z are complex numbers, $wz = 0$ if and only if $w = 0$ or $z = 0$. [*Hint:* write w and z in the form $a + ib$.]

[5 marks]

14. Determine all the (real) values of x such that the series

$$\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n\ (3n-1)}$$

converges.

Explain your reasoning.

[15 marks]

15. By sketching the graphs of $f(x) = \cos x$ and $f(x) = x^2 - 3x - 1$, demonstrate that the equation

$$x^2 - 3x - 1 - \cos x = 0$$

has two solutions (for real x).

Use the Newton-Raphson formula

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

to find the approximate values of both solutions. In each case you are asked to choose an appropriate approximation to the solution, x_0, to use the Newton-Raphson formula to find x_1, and to use x_1 in turn to find another approximation x_2.

In both cases, test whether x_2 is in fact a better approximation to the exact result than x_0.

[15 marks]

16. Let $f(x) = \frac{x^3+3x^2+x+1}{x^2+5x+6}$. Find constants A, B, C, D, E, F such that

$$f(x) = Ax + B + \frac{C}{(x-E)} + \frac{D}{(x-F)}.$$

Given that $f(x)$ has a stationary point for $x \approx 0$, estimate the position of such a point by using the Newton-Raphson method with an initial guess $x_0 = 0$. Perform two iterations of the method and give your result for x_2 with three decimal digits.

[15 marks]

17. Use De Moivre's theorem to show that if $z = \cos\theta + i\sin\theta$ and n is any positive integer then

$$2\cos(n\theta) = z^n + z^{-n} \qquad \text{and} \qquad 2i\sin(n\theta) = z^n - z^{-n}.$$

[*Hint:* try first $n = 1$].

Hence (or otherwise) determine the values of the (real) constants a, b, c, d, e, f for which the following identities hold:

$$(a)\ 2^4\cos^5\theta = a\cos(5\theta) + b\cos(3\theta) + c\cos\theta,$$

$$(b)\ 2^4\sin^5\theta = d\sin(5\theta) + e\sin(3\theta) + f\sin\theta.$$

[15 marks]