2MA1C September 1998

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

Note: $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors pointing in the positive x, y and z directions respectively.

SECTION A

1. Find the values of x which satisfy the equation

$$x = \frac{2}{x+1}.$$

[4 marks]

2. The function f is defined by $f(x) = \frac{3x+2}{x+1}$, $x \neq -1$. Find the inverse function $f^{-1}(x)$ and verify that $f[f^{-1}(x)] = x$. [5 marks]

3. Sketch the graphs in the xy plane represented by the following equations in polar co-ordinates:

(i) $r = \theta, \quad 0 \leq \theta < 2\pi,$ (ii) $r = 1 + \cos\theta, \quad 0 \leq \theta < 2\pi.$

[5 marks]

4. The three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors (relative to the origin O) of the points A, B and C with co-ordinates $A(1, -1, 1)$, $B(3, 2, 6)$ and $C(1, 1, 1)$. Calculate

(i) $\mathbf{a}.\mathbf{b}$, (ii) $\mathbf{b} \times \mathbf{c}$, (iii) $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$.

Hence find the angle between $\mathbf{a}$ and $\mathbf{b}$, the area of the parallelogram with edges parallel to $\mathbf{b}$ and $\mathbf{c}$ and the volume of the parallelepiped with edges parallel to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [7 marks]

5. What is the equation of a plane whose normal is $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and which is a perpendicular distance $2\sqrt{6}$ units from the origin? Verify that this plane passes through the point $(4, -2, 2)$. [5 marks]

6. State L'Hôpital's rule for the evaluation of limits. Hence evaluate

$$\lim_{x \to 0} \frac{\sin^2 x}{\cos x - \cos 3x}.$$

[6 marks]

7. Differentiate the following functions with respect to x:

(i) $\cos^3 x$, (ii) e^{x^2+1}, (iii) $\dfrac{x-1}{x^2+1}$, (iv) $\sinh(x^4)$.

[6 marks]

8. Given that $xy^2 + e^x \sin y = 0$, find $\frac{dy}{dx}$ as a function of x and y. [4 marks]

9. Evaluate the following integrals:

(i) $\displaystyle\int_0^1 xe^{2x}\,dx$, (ii) $\displaystyle\int_0^1 \frac{x^2}{1+x^3}dx$, (iii) $\displaystyle\int_3^5 \frac{x-5}{(x+1)(x-2)}dx$.

[10 marks]

SECTION B

10. Find and classify all stationary points of the function f defined by

$$f(x) = x - 4 + \frac{4}{x+1}, \qquad x \neq -1.$$

Sketch the graph of $y = f(x)$, showing clearly the turning points, asymptotes and the points at which the graph intersects the x and y axes. Show that the tangent to the graph at $x = 3$ crosses the y axis at $y = -\frac{9}{4}$. [16 marks]

11. Find the equation of the plane which passes through the points A, B and C with co-ordinates $A(1, -2, 1)$, $B(2, 2, 2)$ and $C(1, 0, 2)$. Show that the perpendicular distance of this plane from the origin is 2 units.

Find the perpendicular distance of the point $P(1, 1, 1)$ from the plane. A straight line is drawn from P perpendicular to the plane, meeting it at Q. Write down the vector from P to Q, and hence find the co-ordinates of Q. [16 marks]

12. Write down the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}. Hence prove that $2\cosh^2 x - 1 = \cosh 2x$, and show that

$$\cosh 4x + 4\cosh 2x + 3 = 8\cosh^4 x.$$

The inverse hyperbolic sine function $y = \sinh^{-1} x$ is defined by the equation $\sinh y = x$. Show that $e^{2y} - 2xe^y - 1 = 0$, and hence prove that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).$$

Use this formula to verify that $\sinh^{-1} x + \sinh^{-1}(-x) = 0$. [16 marks]

13. A force $\mathbf{F}$ is applied at a point on a rigid body with position vector $\mathbf{r}$ relative to the origin O. Show that the magnitude of the vector $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ gives the magnitude of the turning moment of $\mathbf{F}$ about O and the direction of $\mathbf{M}$ gives the axis of the rotation which $\mathbf{F}$ would produce.

A rigid body contains the points P_1, P_2, and P_3 with cartesian co-ordinates $P_1(1, 2, 3)$, $P_2(2, -1, 1)$ and $P_3(-1, 1, 1)$. Write down the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ of P_1, P_2, and P_3 respectively. A force $\mathbf{F}_1 = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ is applied at P_1 and a force $\mathbf{F}_2 = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$ is applied at P_2. Show that their total turning moment about O is given by

$$(\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) = 2\mathbf{i} + 8\mathbf{j} - 6\mathbf{k}.$$

Additional forces $\mathbf{F}_3 = \lambda(7\mathbf{i}+2\mathbf{j}+5\mathbf{k})$, applied at P_3, and $\mathbf{F}_4$, applied at O, also act on the body. Assuming that the body is maintained in static equilibrium under the action of all four forces, find the value of λ and show that

$$\mathbf{F}_4 = \frac{1}{3}(5\mathbf{i} + 7\mathbf{j} + 10\mathbf{k}).$$

[16 marks]

14. The curve C is the section of the graph $y = \frac{1}{3}x^3$ which lies between $x = 0$ and $x = 1$. Draw carefully a sketch showing C and the straight line $y = \frac{1}{3}x$ on the same diagram. Show that the area enclosed between the curve C and the straight line has the value $\frac{1}{12}$ units.

Show that the volume of the solid generated by rotating the area between C and the x axis through 360° about the x axis is $\frac{1}{63}\pi$ units.

Finally, show that the area S of the curved surface of this solid is given by

$$S = \frac{2}{3}\pi \int_0^1 x^3\sqrt{1 + x^4}\, dx,$$

and hence evaluate S. [16 marks]