Instructions to candidates

Full marks can be obtained for complete answers to FOUR questions. Only the best FOUR answers will be counted.

1. (a) A particle of mass m moves on the x-axis subject to a force acting in the *negative* x-direction of magnitude ma where a is a positive constant. Find the position $x(t)$ of the particle at time t given that at $t = 0$, $x = 0$ and $\dot{x} = u$ where u is a positive constant.

[5 marks]

(b) A particle of mass m and electric charge q moves on the x-axis under the influence of an electric field acting in the positive x-direction of magnitude $E = E_0 \sin \omega t$, where E_0 and ω are positive constants. Show that the position $x(t)$ of the particle satisfies the equation

$$\ddot{x} = \Omega \sin \omega t$$

where $\Omega = qE_0/m$. Find $x(t)$ given that $x(0) = \dot{x}(0) = 0$.

[11 marks]

(c) If the particle in (b) above is also subject to a frictional force of magnitude $\lambda m \dot{x}$, show that its equation of motion can be written

$$\frac{dv}{dt} + \lambda v = \Omega \sin \omega t. \qquad (1)$$

where $v \equiv \dot{x}$. Explain briefly why Eq. (1) is valid whichever direction the particle is moving.

Show (or verify) that the general solution of Eq. (1) is

$$v = \frac{\Omega}{\lambda^2 + \omega^2} (\lambda \sin \omega t - \omega \cos \omega t) + ce^{-\lambda t},$$

and determine the constant c using the initial condition $\dot{x}(0) = 0$.

[9 marks]

[You may use without derivation the result

$$\int e^{\lambda t} \sin \omega t \, dt = \frac{e^{\lambda t}}{\lambda^2 + \omega^2} (\lambda \sin \omega t - \omega \cos \omega t) + c.]$$

2. The equation of motion for a particle of mass m and charge q in a magnetic field $\mathbf{B}$ is

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \wedge \mathbf{B}.$$

If $\mathbf{B} = (0, 0, B)$, where B is a constant, show that this equation becomes:

$$m\ddot{x} = qB\dot{y} \qquad (1a)$$

$$m\ddot{y} = -qB\dot{x} \qquad (1b)$$

$$m\ddot{z} = 0 \qquad (1c)$$

[3 marks]

At $t = 0$ the charge is projected from the origin along the positive y axis with speed u. Show that the particle remains in the (x, y) plane for all time t.

[3 marks]

Integrate Eq. (1a) with respect to time and substitute in Eq. (1b), and hence show that

$$\ddot{y} = -\Omega^2 y,$$

where $\Omega = |qB/m|$.

[5 marks]

Write down the general solution of this equation, and input the initial conditions on y and $\dot{y}$ to find $y(t)$. Using this result, find $\dot{x}(t)$ and hence $x(t)$.

[8 marks]

Show that the resulting trajectory is a circle, and find its radius and where its centre is. Sketch the trajectory.

[6 marks]

3. The motion of a particle of mass m in a central potential $V(r)$ satisfies the following equations:

$$\tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E \tag{1a}$$

$$J = mr^2\dot{\theta} \tag{1b}$$

where J and E are constants; comment briefly on their physical significance.

[2 marks]

Show that

$$\frac{dr}{d\theta} = \frac{mr^2}{J}\dot{r}$$

and hence that:

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{2mr^4}{J^2}\left[E - V\right]. \tag{2}$$

[5 marks]

Now consider two distinct cases:

(a) If the orbit of the particle satisfies the equation

$$r = r_0 e^{k\theta} \tag{3}$$

where r_0 and k are constants, use Eqs. (2) and (3) to find an expression (in terms of r, E, J, m and k) for the potential $V(r)$.

[6 marks]

Show (or verify) using Eqs. (1b) and (3) that if $\theta = 0$ when $t = 0$ then

$$\theta(t) = \frac{1}{2k}\ln\left[1 + \frac{2kJ}{mr_0^2}t\right].$$

[7 marks]

(b) If $V(r) = -\alpha/r^4$, where $\alpha > 0$, verify from Eq. (2) that for $E = 0$ a possible trajectory of the particle is given by

$$r = R_0 \sin\theta$$

where R_0 is a constant. Give an expression for R_0 in terms of J, m and α.

[5 marks]

4. (a) Find the moment of inertia of a uniform circular disc of mass M and radius a, for an axis through its centre and normal to its plane. Then using the Perpendicular Axes theorem, or otherwise, find the moments of inertia of the disc for a pair of mutually perpendicular axes through its centre and in the plane of the disc.

[10 marks]

(b) Explain briefly the meaning of the terms *principal axes* and *principal moments of inertia*. Are the axes described in (a) above principal axes?

[3 marks]

(c) If a body rotates freely about a fixed point, then its angular velocity $\boldsymbol{\omega}$ satisfies the following equations:

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$$
$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1)$$
$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2)$$

Here the components of $\boldsymbol{\omega}$ are taken along the directions of the principal axes and $I_1 \cdots I_3$ are the corresponding principal moments of inertia.

A uniform circular disc, of mass M and radius a, rotates freely with angular velocity $\boldsymbol{\omega}$ about a fixed pivot at its centre. Choosing I_3 to be the moment of inertia about the axis through its centre and normal to its plane, show that ω_3 is a constant and that ω_2 satisfies the equation

$$\ddot{\omega}_2 = -\Omega^2\omega_2$$

where $\Omega = \omega_3$. Hence show that if at $t = 0$ we have $\omega_2 = 0$ then at subsequent t,

$$\boldsymbol{\omega} = (A\cos\Omega t, A\sin\Omega t, \Omega)$$

where A is a constant.

[12 marks]

5. A particle of rest mass m moves on the x axis with relativistic speed v. Write down expressions for its momentum p and its total energy E in terms of m and v. From these expressions, prove that

$$E^2 - p^2c^2 = m^2c^4.$$

[7 marks]

Indicate briefly the significance of this relation with regard to a four-vector involving energy and momentum.

[3 marks]

A particle of rest mass m_1 and total energy E moving along the positive x axis collides with a stationary particle of rest mass m_2. The particles coalesce, becoming a particle of mass M which continues along the x axis.

Show that

$$M^2 = m_1^2 + m_2^2 + \frac{2m_2E}{c^2}.$$

[12 marks]

Calculate M in units of $\mathrm{GeV/c^2}$ given that the particle of mass m_1 is an electron with total energy 30GeV and the particle of mass m_2 is a proton. Give your answer correct to two decimal places.

[3 marks]

[The electron mass is $m_e = 0.511 \times 10^{-3}\mathrm{GeV/c^2}$ and the proton mass is $m_p = 0.938\mathrm{GeV/c^2}$]