MATH5300-01

This question paper consists of 3 printed pages plus a formula sheet, each of which is identified by the reference MATH5300-01

Only approved basic scientific calculators may be used

Examination for the Module MATH5300
(May 2007)
Applied Financial Modelling

Time allowed: 3 hours

Answer 5 questions. All questions are of equal value.
Show your working in answers to all questions.

1. (a) Assume that interest is compounded continuously with rate 7.3%, and that a year has 365 days.

i. If £10 000 000 is deposited into a bank account on 1 June 2007, how much will be in the account on 1 September 2007?

ii. If £10 000 000 is deposited into a bank account on 1 June 2007, on what day will the amount in the account be £10 730 000?

iii. If shares in a company are purchased for £10.00 each on 1 June 2007 and sold for £10.05 each on 21 June 2007, what is the profit per share, taking into account interest rates?

(b) State the conditions that a function $f(x)$ must satisfy in order to be a probability density.

Which of the following is a probability density?

i. $f_{\rm a}(x) = \begin{cases} 0 & x < 0 \\ \mathrm{e}^{-2x} & x \geq 0. \end{cases}$

ii. $f_{\rm b}(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x & x \geq 0. \end{cases}$

iii. $f_{\rm c}(x) = \begin{cases} 0 & x < 0 \\ 3x^2 & 0 \leq x \leq 1 \\ 0 & x > 1. \end{cases}$

iv. $f_{\rm d}(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{16}x^{1/2} & 0 \leq x \leq 4 \\ 0 & x > 4. \end{cases}$

2. (a) If the following is the fair price of a financial derivative,

$$h(\mathbf{S}_t, t) = \mathrm{e}^{\beta t}(C_1 \mathbf{S}_t^{\alpha_1} + C_2 \mathbf{S}_t^{\alpha_2}),$$

what condition must α_1 and α_2 satisfy? (C_1 and C_2 are constants.)

(b) Suppose an investor holds the following portfolio, known as a bearish vertical spread:

- long one European call option
- short another European call option on the same underlying asset, with identical expiry date but lower strike price.

i. By considering three different ranges of the asset price at expiry, sketch the payoff of the portfolio as a function of the asset price on the expiry date.

ii. Write a one-line formula for the payoff.

3. (a) If

$$\mathrm{d}\mathbf{X}_t = -\mu \mathbf{X}_t \mathrm{d}t + \sigma \mathrm{d}\mathbf{W}_t$$

and $\mathbf{R}_t = {\mathbf{X}_t}^2$, what is the stochastic differential equation satisfied by $\mathbf{R}_t$?

(b) The price of a certain type of option on an asset $\mathbf{S}_t$ is

$$\mathbf{U}_t = \begin{cases} E - \mathbf{S}_t & \mathbf{S}_t < \frac{E}{2} \\ \dfrac{E^2}{2\mathbf{S}_t} & \mathbf{S}_t \geq \frac{E}{2}. \end{cases}$$

Assume that $r = 0.045$ and $\sigma = 0.3$.

i. Does this price satisfy the Black-Scholes equation?

ii. From this price, can you deduce what type of option it is? European or American? Call or Put? Sketch a suitable diagram to illustrate.

4. A stochastic process $\mathbf{Z}_t$ is defined by

$$\mathbf{Z}_t = (1+t)^{-2}\,(1 + t + \mathbf{W}_t)\,,$$

where $\mathbf{W}_t$ is the Wiener process.

(a) i. Use Ito's formula to show that $\mathbf{Z}_t$ satisfies the stochastic differential equation

$$\mathrm{d}\mathbf{Z}_t = \left(\frac{1}{(1+t)^2} - \frac{2}{1+t}\mathbf{Z}_t\right)\mathrm{d}t + \frac{1}{(1+t)^2}\mathrm{d}\mathbf{W}_t.$$

ii. Sketch some sample paths of $\mathbf{Z}_t$.

iii. Is it possible that $\mathbf{Z}_t < 0$ at some time t? What happens as $t \to \infty$?

(b) Suppose $\mathbf{S}_t$ satisfies the stochastic differential equation

$$\mathrm{d}\mathbf{S}_t = r\mathbf{S}_t \mathrm{d}t + \sigma \mathbf{S}_t \mathrm{d}\mathbf{W}_t$$

and $\mathbf{Y}_t = \mathrm{e}^{-rt} g(\mathbf{S}_t, t)$, for some function $g(x,t)$.

i. Using Ito's formula, write down the stochastic differential equation that $\mathbf{Y}_t$ satisfies.

ii. If $g(x,t)$ satisfies the Black-Scholes partial differential equation, how does the SDE simplify?

5. A financial derivative has a payoff

$$\mathbf{V}_T = \begin{cases} 2, & \mathbf{S}_T \geq K, \\ 1 & \mathbf{S}_T < K, \end{cases}$$

where $\mathbf{S}_t$ is the underlying asset price at time t and T is the expiry date.

(a) Use risk-neutral valuation to find the fair price for the contract.

(b) Check that your price satisfies the Black-Scholes partial differential equation.

6. Find each of the following quantities, using the formula sheet where necessary. State whether it is positive or negative and what this means in real markets.
Hint: Make good use of the formula sheet.

(a) "Delta", defined as
$$\Delta = \frac{\partial C}{\partial \mathbf{S}_t}.$$

(b) "Gamma", defined as
$$\Gamma = \frac{\partial^2 C}{\partial \mathbf{S}_t^2}.$$

(c) "rho", defined as
$$\rho = \frac{\partial C}{\partial r}.$$

(d) "vega" defined as
$$\text{vega} = \frac{\partial C}{\partial \sigma}.$$

(e) "Theta", defined as
$$\Theta = \frac{\partial C}{\partial t}.$$

(f) The partial derivative
$$\frac{\partial C}{\partial E}.$$

END OF EXAM

Formulas

(1) If the function $f(x)$ is the probability density of the random variable $\mathbf{X}$ then

$$\mathcal{P}[a < \mathbf{X} < b] = \int_a^b f(x)\mathrm{d}x.$$

The mean of the random variable is

$$\mathbb{E}(\mathbf{X}) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x.$$

If $\mathbf{Y} = h(\mathbf{X})$, then the mean of the random variable $\mathbf{Y}$ is

$$\mathbb{E}(\mathbf{Y}) = \int_{-\infty}^{\infty} h(x)f(x)\mathrm{d}x\,.$$

(2) The function $\Phi(x)$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{1}{2}y^2\right)\mathrm{d}y\,.$$

Three properties are $\Phi(-x) = 1 - \Phi(x)$, $\Phi'(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$ and $\Phi''(x) = -x\Phi'(x)$.

(3) The Ito change-of-variables formula. If

$$\mathrm{d}\mathbf{R}_t = a(\mathbf{R}_t, t)\mathrm{d}t + b(\mathbf{R}, t)\mathrm{d}\mathbf{W}_t$$

and

$$\mathbf{V}_t = h(\mathbf{R}_t, t),$$

then

$$\mathrm{d}\mathbf{V}_t = \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}a + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\right)\mathrm{d}t + \left(\frac{\partial h}{\partial x}b\right)\mathrm{d}\mathbf{W}_t\,.$$

(4) The Black-Scholes asset price model:

$$\mathbf{S}_t = \mathbf{S}_0\exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma\mathbf{W}_t\right).$$

(5) The Black-Scholes partial differential equation:

$$\frac{\partial h}{\partial t} + r\mathbf{S}_t\frac{\partial h}{\partial \mathbf{S}_t} + \frac{1}{2}\sigma^2\mathbf{S}_t^2\frac{\partial^2 h}{\partial \mathbf{S}_t^2} - rh = 0.$$

(6) The price of a European call option, with strike price E and expiry date T, is given by

$$C(\mathbf{S}_t, t) = \mathbf{S}_t\Phi(d_1) - \mathrm{e}^{-r(T-t)}E\Phi(d_2)\,.$$

The functions d_1 and d_2 are

$$d_1(\mathbf{S}_t, t) = \frac{\ln(\mathbf{S}_t/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2(\mathbf{S}_t, t) = \frac{\ln(\mathbf{S}_t/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,.$$

Note that

$$\mathbf{S}_t\Phi'(d_1) = \mathrm{e}^{-r(T-t)}E\Phi'(d_2),$$

$$\frac{\partial d_1}{\partial \mathbf{S}_t} = \frac{\partial d_2}{\partial \mathbf{S}_t}, \qquad \frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r}, \qquad \frac{\partial d_1}{\partial \sigma} = -\frac{\partial d_2}{\partial \sigma},$$

$$\frac{\partial d_1}{\partial t} = \frac{1}{2}\frac{\ln(\mathbf{S}_t/E)}{(T-t)^{\frac{3}{2}}} - \frac{1}{2}\frac{r + \frac{1}{2}\sigma^2}{\sigma}\frac{1}{\sqrt{T-t}} = -\frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} + \frac{1}{2}\frac{d_1}{T-t}$$

and

$$\frac{\partial d_2}{\partial t} = \frac{1}{2}\frac{\ln(\mathbf{S}_t/E)}{(T-t)^{\frac{3}{2}}} - \frac{1}{2}\frac{r - \frac{1}{2}\sigma^2}{\sigma}\frac{1}{\sqrt{T-t}} = -\frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} + \frac{1}{2}\frac{d_2}{T-t}.$$