MATH5300-01

This question paper consists of 2 printed pages plus a formula sheet, each of which is identified by the reference MATH5300-01

Only approved basic scientific calculators may be used

Examination for the Module MATH5300
(May 2006)
Applied Financial Modelling

Time allowed: 3 hours

Attempt no more than 5 questions. All questions are of equal value.
Show your working in answers to all questions.

1. (a) How much must be deposited in a bank account on 1 August 2006, in order to have £100,000 in the account on 7 August 2007? Assume that interest is compounded continuously with rate 9%, and that a year has 365 days.

(b) Let $\mathbf{G}$ be a Gaussian random variable with mean 0 and variance 1.

i. If $\mathbf{X} = 3.2\mathbf{G}$, find the mean and variance of $\mathbf{X}$.

ii. If $\mathbf{Y} = \exp(-1.5\mathbf{G})$, find the mean and variance of $\mathbf{Y}$.

iii. If $\mathbf{Z} = 3\mathbf{G}^2 + 1$, find the mean and variance of $\mathbf{Z}$. *Hint:* $\mathbb{E}(\mathbf{G}^4) = 3$

(c) The price of the stock ORA at time t is $\mathbf{S}_t$ where

$$\mathbf{S}_t = 100 \exp\left(5t + 2\mathbf{W}_t\right) ,$$

and $\mathbf{W}$ is the Wiener process.

i. Find $\ln(\mathbb{E}(\mathbf{S}_t))$.

ii. Find $\mathbb{E}(\ln \mathbf{S}_t)$.

iii. If $\mathbf{S}_{1.5} = 150$, find $\mathbb{E}(\mathbf{S}_{1.75})$.

2. (a) The random variable $\mathbf{Y}$ is defined as $\mathbf{Y} = \exp(t^2 + \frac{1}{2}\sqrt{t}\mathbf{G})$, where $\mathbf{G}$ is a Gaussian random variable with mean zero and variance 1. Calculate $\mathbb{E}(\sqrt{\mathbf{Y}})$.

(b) The stochastic process $\mathbf{Z}_t$ is defined as $\mathbf{Z}_t = \ln(1 + {\mathbf{W}_t}^2)$, where $\mathbf{W}$ is the Wiener process. Find the stochastic differential equation for $\mathbf{Z}_t$.

(c) The fair price of a financial derivative is

$$h(\mathbf{S}_t, t) = 63\mathrm{e}^{-r(T-t)}\mathbf{S}_t^{\alpha}.$$

What values of α are allowed under the Black-Scholes theory?

3. Let $\mathbf{U}_t = \exp(\sqrt{t} + {\mathbf{W}_t}^4)$, where $\mathbf{W}$ is the Wiener process. and $t > 0$.

(a) Write down the integral that gives $\mathcal{P}[\mathbf{U}_t < x]$ as a function of x.

(b) Find the probability density of $\mathbf{U}_t$.

(c) Find the stochastic differential equation for $\mathbf{U}_t$.

(d) Sketch some sample trajectories of $\mathbf{U}_t$.

4. (a) A "collar option" has a payoff

$$\mathbf{C}_T = 2\min\left(\max(\mathbf{S}_T, K_1), K_2\right),$$

where K_1 and K_2 are real numbers with $0 < K_1 < K_2$. Sketch the payoff diagram as a function of $\mathbf{S}_T$.

Hint: consider the three cases $\mathbf{S}_T < K_1$, $K_1 \le \mathbf{S}_T \le K_2$ *and* $\mathbf{S}_T > K_2$.

(b) Describe the differences between a discrete random variable and a continuous random variable. Give examples of each. How are probabilities assigned?

(c) The function

$$V(\mathbf{S}_t, t) = E\mathrm{e}^{-r(T-t)}(1 - \Phi(b(\mathbf{S}_t, t))),$$

where

$$b(\mathbf{S}_t, t) = \frac{(\frac{1}{2}\sigma^2 - r)(T-t) + \ln K - \ln \mathbf{S}_t}{\sigma\sqrt{T-t}},$$

and E and K are constants, is the fair price of a financial derivative with expiry date T.

What is the payoff of the contract?

5. (a) Using the Black-Scholes asset price model, show that the probability that a European put option, with strike price E at expiry date T on an asset whose price at time $t < T$ is $\mathbf{S}_t$, is exercised is equal to the probability that

$$\mathbf{G} > b(\mathbf{S}_t, t),$$

where $\mathbf{G}$ is a Gaussian random variable with mean zero and variance 1, and

$$b(\mathbf{S}_t, t) = \frac{\ln(\mathbf{S}_t/E) + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

(b) A "digital put" option has a payoff

$$\mathbf{P}_T = \begin{cases} 0, & \mathbf{S}_T \ge K, \\ 10, & \mathbf{S}_T < K, \end{cases}$$

where $\mathbf{S}_t$ is the underlying asset price at time t.

i. Use risk-neutral valuation to show that the fair price for the contract for $t < T$ is

$$P(\mathbf{S}_t, t) = 10\mathrm{e}^{-r(T-t)}(1 - \Phi(d(\mathbf{S}_t, t))),$$

and find $d(\mathbf{S}_t, t)$. *Hint: use the answer to part (a)*

ii. Check that $P(\mathbf{S}_t, t)$ satisfies the Black-Scholes partial differential equation. *Hint: use the properties of* $\Phi(x)$ *on the formula sheet.*

6. A financial institution offers a contract that pays an amount equal to $(\ln \mathbf{S}_T)^2$ at time T, where $\mathbf{S}_t$ is the underlying asset price at time t.

(a) Use risk-neutral valuation to calculate the fair price for the contract for $t < T$.

(b) Check that your answer satisfies the Black-Scholes partial differential equation.

END OF EXAM

Formulas

(1) Put-call parity:

$$\mathbf{C}_t + E\mathrm{e}^{-r(T-t)} = \mathbf{P}_t + \mathbf{S}_t \,.$$

(2) If the function $f(x)$ is the probability density of the random variable $\mathbf{X}$ then

$$\mathcal{P}[a < \mathbf{X} < b] = \int_a^b f(x)\mathrm{d}x.$$

The mean of the random variable is

$$\mathbb{E}(\mathbf{X}) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x.$$

For a Gaussian random variable with mean μ and standard deviation σ,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/2\sigma^2\right) .$$

If $\mathbf{Y} = h(\mathbf{X})$, then the mean of the random variable $\mathbf{Y}$ is

$$\mathbb{E}(\mathbf{Y}) = \int_{-\infty}^{\infty} h(x) f(x)\mathrm{d}x \,.$$

(3) If $\mathbf{G}$ is a Gaussian random variable with mean 0 and variance 1 then $\mathbb{E}(\exp(m + s\mathbf{G})) = \mathrm{e}^{m+\frac{1}{2}s^2}$.

(4) The function $\Phi(x)$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}y^2\right)\mathrm{d}y \,.$$

Three properties are

$$\Phi(-x) = 1 - \Phi(x),$$

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

and

$$\Phi''(x) = -x\Phi'(x).$$

(5) The Ito change-of-variables formula. If

$$\mathrm{d}\mathbf{R}_t = a(\mathbf{R}_t, t)\mathrm{d}t + b(\mathbf{R}, t)\mathrm{d}\mathbf{W}_t$$

and

$$\mathbf{V}_t = h(\mathbf{R}_t, t),$$

then

$$\mathrm{d}\mathbf{V}_t = \left(\frac{\partial h}{\partial x}a + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\right)\mathrm{d}t + \left(\frac{\partial h}{\partial x}b\right)\mathrm{d}\mathbf{W}_t \,.$$

(6) The Black-Scholes asset price model:

$$\mathbf{S}_t = \mathbf{S}_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma\mathbf{W}_t\right) .$$

(7) The Black-Scholes partial differential equation:

$$\frac{\partial h}{\partial t} + \frac{1}{2}\sigma^2\mathbf{S}_t^2\frac{\partial^2 h}{\partial \mathbf{S}_t^2} + r\mathbf{S}_t\frac{\partial h}{\partial \mathbf{S}_t} - rh = 0.$$