MATH5300-01

This question paper consists of 2 printed pages plus a formula sheet, each of which is identified by the reference MATH5300-01

Only approved basic scientific calculators may be used

Examination for the Module MATH5300
(May 2004)

Applied Financial Modelling

Time allowed: 3 hours

Attempt no more than 5 questions. All questions are of equal value.

Show your working in answers to all questions.

1. Assume that interest is compounded continuously with interest rate 5%.

(a) i. If £1000 is deposited in a bank account on 1 August 2004, how much will be in the account on 1 August 2005?

ii. If £1000 is deposited in a bank account on 1 August 2004, how much will be in the account on 1 September 2005?

iii. How much must be deposited in a bank account on 1 August 2004, in order to have £2000 1 August 2006?

(b) The price of a European call option is £1.3231. Its expiry date is 1 year from now and its strike price is £4. Given that the current price of the stock is £5, use put-call parity to find the current price of a put option on the same stock.

2. (a) If $\mathbf{W}_t$ is the Wiener process,

i. Write the probability density of the random variable $\mathbf{W}_{t_2} - \mathbf{W}_{t_1}$.

ii. If $\mathbf{W}_3 = -1$, write the probability density of the random variable $\mathbf{W}_4$.

iii. If $\mathbf{Y}_0 = 0$ and $\mathbf{Y}_t = \alpha t + \beta \mathbf{W}_t$, where α and β are constants, write the probability density of $\mathbf{Y}_t$.

(b) If $\mathbf{G}$ is a Gaussian random variable with mean zero and variance 1, and

$$\mathbf{X} = \ln(\mathbf{G}^2),$$

find the probability density of $\mathbf{X}$.

3. The price of the stock XYZ at time t is $\mathbf{S}_t$ where

$$\mathbf{S}_t = \mathbf{S}_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma \mathbf{W}_t \right) .$$

(a) Write down the formula for $\mathbf{S}_t^2$. Find $\mathbb{E}(\mathbf{S}_t^2)$.
Hint: Use (3) from the formula sheet.

(b) A financial institution plans to offer a derivative with payoff equal to $\mathbf{S}_T^2$. Use risk-neutral valuation to calculate the fair price for this derivative at any time $0 \le t \le T$. Check that your answer satisfies the Black-Scholes partial differential equation.

4. (a) Find a value of n such that

$$h(\mathbf{S}_t, t) = \mathbf{S}_t^n \mathrm{e}^{(\sigma^2 - 2r)(T-t)}$$

is the fair price of a financial derivative with expiry date T, where $\mathbf{S}_t$ is the price of the underlying asset at time t. What is its payoff?

(b) Find the values of the constants A and B such that the function

$$U(x,t) = \frac{A}{\sqrt{t}} \exp\left(-\frac{x^2}{Bt}\right)$$

is a solution of

$$\frac{\partial}{\partial t} U(x,t) = \frac{\partial^2}{\partial x^2} U(x,t).$$

5. (a) Let $\mathbf{V}_t = {\mathbf{W}_t}^2$. Find the stochastic differential equation for $\mathbf{V}_t$.

(b) Let $\mathbf{Z}_t = \exp(\mathbf{W}_t - \alpha t)$. For which value of α is $\mathbb{E}(\mathbf{Z}_t)$ constant in time?

6. (a) The fair price of a European call option is $C(\mathbf{S}_t, t)$ where

$$C(S,t) = S\Phi(d_1) - E\mathrm{e}^{-r(T-t)}\Phi(d_2)\ ,$$

$$d_1 = \frac{\ln(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = \frac{\ln(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\ .$$

Use put-call parity to find the fair price of a European put option.

(b) Explain why the price of an American call option is the same as that of a European call option.

7. (a) Show that, in the Black-Scholes theory, the probability that a European call option is exercised, given that the expiry date is T, the strike price is K and the asset price at time t is $\mathbf{S}_t$, is

$$\mathcal{P}[\text{option exercised}] = \Phi(b(\mathbf{S}_t, t))\,,$$

where

$$b(x,t) = \frac{(\mu - \frac{1}{2}\sigma^2)(T-t) + \ln\frac{x}{K}}{\sigma\sqrt{T-t}}.$$

(b) A financial institution offers a contract whose payoff is equal to £100 if $\mathbf{S}_T > K$ and is equal to zero otherwise. Use risk-neutral valuation to find a fair price for the contract. Check that your answer satisfies the Black-Scholes partial differential equation.

END OF EXAM

Formulas

(1) Put-call parity:

$$C + E\mathrm{e}^{-rT} = P + S\,.$$

(2) If the function $f(x)$ is the probability density of the random variable $\mathbf{X}$ then

$$\mathcal{P}[a < \mathbf{X} < b] = \int_a^b f(x)\mathrm{d}x.$$

The mean of the random variable is

$$\mathbb{E}(\mathbf{X}) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x.$$

For a Gaussian random variable with mean μ and standard deviation σ,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/2\sigma^2\right)\,.$$

If $\mathbf{Y} = h(\mathbf{X})$, then the mean of the random variable $\mathbf{Y}$ is

$$\mathbb{E}(\mathbf{Y}) = \int_{-\infty}^{\infty} h(x) f(x)\mathrm{d}x\,.$$

(3) If $\mathbf{G}$ is a Gaussian random variable with mean 0 and variance 1 then $\mathbb{E}(\exp(m + s\mathbf{G})) = \mathrm{e}^{m+\frac{1}{2}s^2}$.

(4) The function $\Phi(x)$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}y^2\right)\mathrm{d}y\,.$$

$$\Phi(-x) = 1 - \Phi(x)\,.$$

(5) The Ito change-of-variables formula. If

$$\mathrm{d}\mathbf{R}_t = a(\mathbf{R}_t, t)\mathrm{d}t + b(\mathbf{R}, t)\mathrm{d}\mathbf{W}_t$$

and

$$\mathbf{V}_t = h(\mathbf{R}_t, t),$$

then

$$\mathrm{d}\mathbf{V}_t = \left(\frac{\partial h}{\partial x}a + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\right)\mathrm{d}t + \left(\frac{\partial h}{\partial x}b\right)\mathrm{d}\mathbf{W}_t\,.$$

(6) The Black-Scholes asset price model:

$$\mathbf{S}_t = \mathbf{S}_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma\mathbf{W}_t\right)\,.$$

(7) The Black-Scholes partial differential equation:

$$\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial \mathbf{S}_t^2}\sigma^2\mathbf{S}_t^2 + r\mathbf{S}_t\frac{\partial h}{\partial \mathbf{S}_t} - rh = 0.$$