MATH–5031M01

This question paper consists of 5 printed pages, each of which is identified by the reference MATH–5031M01

Only approved basic scientific calculators may be used.

Examination for the Module MATH–5031M

(January 2007)

Differential Geometry 2

Time allowed: 3 hours

Answer a maximum of **four** questions from Section A and a maximum of **two** questions from Section B. All questions carry equal marks.

SECTION A

Throughout Section A, by 'surface' we shall mean 'smooth regular embedded m-surface in $\mathbb{R}^n$ for some positive integers m and n'.

1. **(a)** Let $\gamma : [0,b] \to \mathbb{R}^2$ be a smooth unit speed parametrized curve and let $s \in [0,b]$. What is meant by (i) the *unit positive tangent* of γ at s, (ii) the *unit positive normal* of γ at s, the *signed curvature* $\kappa(s)$ of γ at s?

Let $\theta : [0,b] \to \mathbb{R}$ be a smooth function such that $\gamma'(s) = \big(\cos\theta(s), \sin\theta(s)\big)$ $\big(s \in [0,b]\big)$. Show that $\kappa(s) = \theta'(s)$ $\big(s \in [0,b]\big)$.

Now assume that γ is closed. Show that the total curvature $\int_0^b \kappa(s)\,\mathrm{d}s$ of γ is given by $\theta(b) - \theta(0)$, and define the *rotation index* of γ (you need not show that it is an integer). What is meant by the rotation index of a smooth closed regular parametrized curve which is not of unit speed?

(b) Let $\gamma(t) = (r\sin bt, r\cos bt)$ $\big(t \in [0,2\pi]\big)$, where r and b are positive real constants. Calculate the rotation index of γ.

(c) By using part (b) and finding a suitable regular homotopy, or otherwise, find the rotation index of the closed curve $\alpha : [0,2\pi] \to \mathbb{R}^2$ given by

$$\alpha(t) = \big(4\sin 3t - \sin t\,,\, 4\cos 3t - \cos t\big) \qquad \big(t \in [0,2\pi]\big)\,.$$

2. **(a)** Let n and k be positive integers with $k \leq n$ and let $f : W \to \mathbb{R}^k$ be a smooth map from an open subset W of $\mathbb{R}^n$ to $\mathbb{R}^k$. Say what is meant by (i) $p \in W$ is a *regular point* of f, (ii) $p \in W$ is a *critical point* of f, (iii) $c \in f(W)$ is a *regular value* of f.

Define a smooth mapping $f : \mathbb{R}^4 \to \mathbb{R}^2$ by $f(x_1, x_2, x_3, x_4) = ({x_1}^2 + {x_2}^2,\, 2x_3 - x_4)$. Find the critical points of f and show that $(1, 0)$ is a regular value. Deduce that

$$M = \{(x_1, x_2, x_3, x_4) : {x_1}^2 + {x_2}^2 = 1,\; x_4 = 2x_3\}$$

is a smooth regular embedded 2-surface in $\mathbb{R}^4$.

(b) Let $X : \mathbb{R}^2 \to \mathbb{R}^4$ be the smooth map defined by $X(u_1, u_2) = (\cos u_1\,,\, \sin u_1\,,\, u_2\,,\, 2u_2)$. Show that X is regular and has image the 2-surface M above. Show that X is not injective on $\mathbb{R}^2$. Find a maximal open subset of $\mathbb{R}^2$ on which it is injective.

3. **(a)** Let $f : M \to M'$ be a smooth map between surfaces, and let $p \in M$. Define the *differential* $\mathrm{d}f_p : T_pM \to T_{f(p)}M'$ of f at p. Let $g : M' \to M''$ be another smooth map between surfaces and let $p \in M$. Show that $\mathrm{d}(g \circ f)_p = \mathrm{d}g_{f(p)} \circ \mathrm{d}f_p$.

(b) Let $f : M \to M'$ be a smooth map between surfaces. Define what is meant by f is a (i) *local isometry*, (ii) *global isometry*.

What is meant by the *distance* $d(p, q)$ between two points p and q on a surface M?

Define regular maps $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ by $\phi(u, v) = (\sinh v \sin u, -\sinh v \cos u, u)$ and $\psi(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$. Let H denote the image of ϕ and C the image of ψ. Define a map $f : H \to C$ by

$$f\big(\phi(u, v)\big) = \psi(u, v) \qquad \big((u, v) \in \mathbb{R}^2\big).$$

Show that this is (i) well defined, (ii) a local isometry, (iii) not a global isometry. Show that there are points $p, q \in H$ such that $d\big(f(p), f(q)\big) \neq d(p, q)$.

[You may use the fact that a smooth map $f : M \to M'$ is a local isometry if and only if, for each $p \in M$, there is a basis $\{\mathbf{e}_i\}$ such that $\mathrm{d}f_p(\mathbf{e}_i) \cdot \mathrm{d}f_p(\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j \quad (i, j = 1, 2)$.]

4. **(a)** Let $f : M \to M'$ be a smooth map between 2-surfaces. Say what is meant by f *is conformal with scale factor* λ. Show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if

$$\mathrm{d}f_p(\mathbf{v}) \cdot \mathrm{d}f_p(\mathbf{w}) = \lambda(p)^2\, \mathbf{v} \cdot \mathbf{w} \qquad (p \in M,\ \mathbf{v}, \mathbf{w} \in T_pM)\,.$$

Hence show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if, for all $p \in M$, there exists a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of T_pM such that $|\mathrm{d}f_p(\mathbf{e}_i)| = \lambda(p)\,|\mathbf{e}_i|\ \ (i = 1, 2)$ and $\mathrm{d}f_p(\mathbf{e}_1) \cdot \mathrm{d}f_p(\mathbf{e}_2) = \lambda(p)^2\, \mathbf{e}_1 \cdot \mathbf{e}_2$.

(b) Let $S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$. Define a smooth map $f : \mathbb{R}^2 \to S^2$ by

$$f(x, y) = \frac{1}{1 + x^2 + y^2}\,(2x\,,\ 2y\,,\ 1 - x^2 - y^2) \qquad \big((x, y) \in \mathbb{R}^2\big).$$

Show that ϕ is conformal and determine its scale factor $\lambda : \mathbb{R}^2 \to (0, \infty)$.

Show that f does not map any non-constant parametrized geodesic in $\mathbb{R}^2$ to a parametrized geodesic on S^2.

5. **(a)** Let M be a 2-surface in $\mathbb{R}^3$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis for the tangent space at a point p of M. Write $E = \mathbf{e}_1 \cdot \mathbf{e}_1$, $F = \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_1$, $G = \mathbf{e}_2 \cdot \mathbf{e}_2$, $L = S(\mathbf{e}_1) \cdot \mathbf{e}_1$, $M = S(\mathbf{e}_1) \cdot \mathbf{e}_2 = S(\mathbf{e}_2) \cdot \mathbf{e}_1$, $N = S(\mathbf{e}_2) \cdot \mathbf{e}_2$, where S is the shape operator of M at p. Show that the Gauss curvature $K(p)$ of M at p is given by

$$K(p) = \frac{LN - M^2}{EG - F^2}\,.$$

(b) Let M be a 2-surface in $\mathbb{R}^3$. What is meant by a property is (A) *intrinsic*, (B) *extrinsic*. State *two* extrinsic properties of a surface. (You need not justify your answer.)

State the *Theorema Egregium* of Gauss.

(c) Let M be a closed 2-surface in $\mathbb{R}^3$. Explain briefly what is meant by the (i) *total curvature*, (ii) the *Euler characteristic* of M. (You need not define what is meant by a triangulation or show that the Euler characteristic is well defined.)

State the *Gauss–Bonnet Theorem.*

Use the Gauss–Bonnet Theorem to show that the Gauss curvature of a closed orientable 2-surface of genus 2 cannot be identically zero.

continued ...

SECTION B

6. **(a)** Let $\alpha : I \to \mathbb{R}^3$ be a unit speed smooth closed curve. Define a map $f : I \to S^2$ to the unit sphere S^2 by $f(s) = \alpha'(s) \quad (s \in I)$ and let Γ denote its image. Show that (i) for any $s \in I$, the unsigned curvature $\kappa(s)$ of α at s is equal to the speed of f at s; (ii) the total curvature $\int_I |\kappa(s)| \, \mathrm{d}s$ of α is equal to the length of Γ (i.e., of f).

Now let $\mathbf{a}$ be a fixed unit vector. Define a function $g : I \to \mathbb{R}$ by $g(s) = \mathbf{a} \cdot \alpha(s)$. Show that, if $s \in I$ is a point where g attains a maximum or minimum, then $\mathbf{a} \cdot f(s) = 0$. Deduce that (i) Γ is met by every great circle of S^2, (ii) the length of Γ is at least 2π. Hence show that the total curvature of α is at least 2π.

[You may assume that, if Γ is a smooth closed curve in S^2 of length less than 2π, then there is a point $m \in S^2$ such that the spherical distance of m from x is less than $\pi/2$ for all points $x \in \Gamma$.]

(b) Give an example to show that the total curvature of a unit speed smooth closed curve can be exactly 2π.

7. Let M be an m-surface in $\mathbb{R}^n$. Let $\alpha : [a, b] \to M$ be a unit speed smooth curve on M. For any positive real number p, set

$$E_p(\alpha) = \frac{1}{2} \int_a^b |\alpha'(t)|^p \, \mathrm{d}t \, .$$

(i) Find the first variation formula for E_p.

(ii) Show that, if α is a geodesic, then E_p is stationary with respect to variations which fix the endpoints.

(iii) Show the converse, i.e., if E_p is stationary with respect to variations which fix the endpoints, then α is a geodesic. [You may assume that (A) for any closed interval $[a, b]$ and any $t_0 \in [a, b]$, there are a_1, b_1 with $a < a_1 < t_0 < b_1 < b$ and a smooth function $f : [a, b] \to [0, \infty)$ with $f(t_0) > 0$ and $f(t) = 0$ for all $t \notin [a_1, b_1]$; (B) given a vector field v along α with $v(a) = v(b)$, there is a variation of α with variation vector field v which fixes the endpoints.]

(iv) Show that, if $p = 1$, then (ii) and (iii) hold for a smooth curve $\alpha : [a, b] \to M$ which is not necessarily of unit speed or regular.

continued . . .

8. Let M be a 2-surface in $\mathbb{R}^3$. Let $X : U \to M$, $(u, v) \mapsto X(u, v)$ be a local parametrization of M and let $\mathbf{N}$ denote the corresponding unit positive normal. Let $\epsilon_1 = \partial X/\partial u$ and $\epsilon_2 = \partial X/\partial v$ be the corresponding coordinate vectors. Write $E = \epsilon_1 \cdot \epsilon_1$, $F = \epsilon_1 \cdot \epsilon_2 = \epsilon_2 \cdot \epsilon_1$, $G = \epsilon_2 \cdot \epsilon_2$, $L = S(\epsilon_1) \cdot \epsilon_1$, $M = S(\epsilon_1) \cdot \epsilon_2 = S(\epsilon_2) \cdot \epsilon_1$, $N = S(\epsilon_2) \cdot \epsilon_2$, where S is the shape operator of M at p . Further write

$$\begin{aligned}
X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L\mathbf{N}\,, \\
X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M\mathbf{N}\,, \\
X_{vu} &= \Gamma^1_{21} X_u + \Gamma^2_{21} X_v + M\mathbf{N}\,, \\
X_{vv} &= \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N\mathbf{N}\,.
\end{aligned}$$

Show that the functions Γ^k_{ij} can be expressed in terms of E, F, G and their first order partial derivatives with respect to u and v, and find the expressions in the case that $F = 0$.

Hence show that the Gauss curvature K of M is expressible in terms of the functions E, F, G and their first and second order partial derivatives with respect to u and v.

[You need not find the expression for K. You may assume that K is given by $K = (LN - M^2)\big/(EG - F^2)$, but you should explain briefly why $EG - F^2$ is non-zero.]

END