MATH–5031M01

This question paper consists of 5 printed pages, each of which is identified by the reference MATH–5031M01

Only approved basic scientific calculators may be used.

Examination for the Module MATH–5031M

(January 2006)

Differential Geometry 2

Time allowed: 3 hours

Answer a maximum of **four** questions from Section A and a maximum of **two** questions from Section B. All questions carry equal marks.

Throughout this paper, by 'surface' we shall mean 'smooth regular embedded m-surface in $\mathbb{R}^n$ for some positive integers m and n'.

SECTION A

1. **(a)** Let $\gamma : [0, b] \to \mathbb{R}^2$ be a regularly parametrized curve. What is meant by saying that γ *is closed*? What is meant by the *total curvature of a regularly parametrized closed curve*?

Let $\gamma_0 : [0, b_0] \to \mathbb{R}^2$ and $\gamma_1 : [0, b_1] \to \mathbb{R}^2$ be regularly parametrized closed curves. What is meant by a *regular homotopy from γ_0 to γ_1*? State the *Whitney–Graustein Theorem*.

(b) Let $\gamma(t) = (4\sin 2t, 4\cos 2t) \quad \big(t \in [0, 5\pi]\big)$. Calculate the total curvature of γ.

(c) By finding a suitable regular homotopy and quoting the Whitney–Graustein Theorem, or otherwise, find the total curvature of the closed curve $\alpha : [0, 5\pi] \to \mathbb{R}^2$ given by

$$\alpha(t) = \big(4\sin 2t - 2\sin t\,,\ 4\cos 2t + 2\cos t\big)\,.$$

Show that

$$H(u, t) = \big(4u\sin 2t - 2\sin t\,,\ 4u\cos 2t + 2\cos t\big) \qquad \big(u \in [0, 1],\ t \in [0, 5\pi]\big)$$

does *not* define a regular homotopy.

2. **(a)** Let $\varphi : M \to M'$ be a smooth map between surfaces, and let $p \in M$. Define what is meant by the *differential* $\mathrm{d}\varphi_p : T_pM \to T_{\varphi(p)}M'$ *of* φ *at* p. Let

$$X : U \to M\,, \qquad \mathbf{u} = (u_1, \ldots, u_m) \mapsto X(\mathbf{u})$$

be a local parametrization of M with $X(\mathbf{0}) = p$. Write $\hat{\varphi} = \varphi \circ X$ and $\epsilon_i = \partial X / \partial u_i$ $(i = 1, \ldots, m)$. Show that

$$\mathrm{d}\varphi_p(\epsilon_i) = \frac{\partial \hat{\varphi}}{\partial u_i}(\mathbf{0}) \qquad (i = 1, \ldots, m)\,.$$

Deduce that, if $\mathbf{v} \in T_pM$ is given by $\mathbf{v} = \sum_{i=1}^m v_i\,\epsilon_i$, then

$$\mathrm{d}\varphi_p(\mathbf{v}) = \sum_{i=1}^m v_i\,\mathrm{d}\varphi_p(\epsilon_i)\,.$$

(b) Let $f : M \to M'$ be a smooth map between surfaces. Define what is meant by f is a *local isometry.*

Show that, if f is a local isometry then,

(*) *for any smooth curve* $\alpha : [a, b] \to M$ *defined on a closed interval* $[a, b]$*, the length of* $f \circ \alpha$ *is equal to the length of* α*.*

Show conversely that, if $f : M \to M'$ is a smooth map having the property (*), then it is a local isometry.

3. **(a)** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the smooth function $f(x_1, x_2, x_3) = x_1^{\,2} + x_2^{\,2} + x_2^{\,2}$. For $r > 0$, set $S^2(r) = f^{-1}(r)$. Show that f is regular on $S^2(r)$, so that $S^2(r)$ is a 2-surface. Without parametrizing $S^2(r)$, show that its shape operator S at any point $p \in S^2(r)$ is given by

$$S(\mathbf{v}) = c\,\mathbf{v} \qquad \big(\mathbf{v} \in T_pS^2(r)\big)$$

for some constant c to be determined.

(b) Let M be a surface and let $\gamma : I \to M$ be a smooth curve defined on an interval I. Say what is meant by γ *is a geodesic on* M. Show that the speed $|\gamma'(t)|$ $(t \in I)$ of a geodesic is constant.

(c) Suppose that $\gamma : I \to S^2(r)$ is a geodesic of unit speed. Show that it is a plane curve of constant curvature $1/r$; deduce that its track lies on a great circle of $S^2(r)$.

Suppose, instead, that $\gamma : I \to S^2(r)$ is a smooth curve of unit speed whose principal normal makes a constant angle with a unit normal of $S^2(r)$. Show that the track of γ lies on a circle and give the radius of that circle.

[You may assume that the track of a unit speed plane curve of constant curvature $1/r$ lies on a circle of radius r.]

4. (a) Let $f : M \to M'$ be a smooth map between 2-surfaces. Say what is meant by f *is conformal with scale factor* λ. Show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if

$$\mathrm{d}f_p(\mathbf{v}) \cdot \mathrm{d}f_p(\mathbf{w}) = \lambda(p)^2\, \mathbf{v} \cdot \mathbf{w} \qquad (p \in M,\ \mathbf{v}, \mathbf{w} \in T_pM)\,.$$

Give a formula for the angle between two non-zero vectors, and show that a smooth map $f : M \to M'$ is conformal if and only if it preserves angles in the sense that, for all $p \in M$ and all non-zero $\mathbf{v}, \mathbf{w} \in T_pM$, the vectors $\mathrm{d}f_p(\mathbf{v})$ and $\mathrm{d}f_p(\mathbf{w})$ are non-zero and the angle between them is equal to the angle between $\mathbf{v}$ and $\mathbf{w}$.

(b) Let $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and

$$E^2 = \left\{(x, y, z) : \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1\right\}$$

where a and b are positive constants. Define a smooth map $f : S^2 \to E^2$ by

$$f(x, y, z) = (ax, ay, bz)\,.$$

Show that f is conformal if and only if $a = b$. Determine the scale factor of f in this case.

5. (a) Give a formula which defines a local isometry from the plane $\mathbb{R}^2$ to the unit circular cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. [You need not show that this is a local isometry.]

(b) Let M be a 2-surface in $\mathbb{R}^3$. What is meant by saying that a property is (A) *intrinsic*, (B) *extrinsic*. Show that the following properties are extrinsic: (i) principal curvatures; (ii) mean curvature; (iii) distance between pairs of points [You may quote the values of the principal curvatures of $\mathbb{R}^2$ in $\mathbb{R}^3$ and of C in $\mathbb{R}^3$ without proof.] State the *Theorema Egregium* of Gauss.

(c) Let M be a closed 2-surface. Explain briefly what is meant by the (i) *total curvature* of M, (ii) *Euler characteristic* of M. [You need not define what is meant by a triangulation or show that the Euler characteristic is well defined.] State the *Gauss–Bonnet Theorem.*

Let M be a closed 2-surface with Euler characteristic 0 and Gauss curvature K satisfying $K \leq 0$ at all points. Show that K is identically zero.

continued . . .

SECTION B

6. **(a)** Prove that, for any nonnegative real numbers a and b,

$$\sqrt{ab} \leq \frac{1}{2}(a+b)\,.$$

Prove that, for any four real numbers a_1, a_2, b_1, b_2,

$$\left(\sum_{i=1}^{2} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{2} a_i^2\right)\left(\sum_{i=1}^{2} b_i^2\right).$$

(b) Let C: $s \mapsto \mathbf{x}(s) = (x_1(s), x_2(s))$ $(s \in [0, L])$ be a positively oriented unit speed simple closed curve in the plane of length L which encloses an area A. Describe how to construct a circle $\overline{C}$: $s \mapsto \overline{\mathbf{x}}(s) = (\overline{x}_1(s), \overline{x}_2(s))$ $(s \in [0, L])$ with $\overline{x}_1(s) = x_1(s)$ $(s \in [0, L])$. Show that

$$A + \pi r^2 \leq Lr\,.$$

where r is the radius of the circle. Deduce the isoperimetric inequality:

$$4\pi A \leq L^2.$$

7. Let M be a 2-surface in $\mathbb{R}^3$ and let $X : U \to M$, $(u, v) \mapsto X(u, v)$ be a local parametrization of M. As usual, write $\epsilon_1 = \partial X/\partial u$ and $\epsilon_2 = \partial X/\partial v$, $E = \epsilon_1 \cdot \epsilon_1$, $F = \epsilon_1 \cdot \epsilon_2 = \epsilon_2 \cdot \epsilon_1$, $G = \epsilon_2 \cdot \epsilon_2$, $L = S(\epsilon_1) \cdot \epsilon_1$, $M = S(\epsilon_1) \cdot \epsilon_2 = S(\epsilon_2) \cdot \epsilon_1$, $N = S(\epsilon_2) \cdot \epsilon_2$, where S is the shape operator of M at p.

(a) Show that the Gauss curvature K of M at a point in the image of X is given by

$$K = \frac{LN - M^2}{EG - F^2}\,.$$

(b) Write

$$\begin{aligned} X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L\,\mathbf{N} \\ X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M\mathbf{N} \\ X_{vu} &= \Gamma^1_{21} X_u + \Gamma^2_{21} X_v + M\mathbf{N} \\ X_{vv} &= \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N\mathbf{N} \end{aligned}$$

Suppose that F is identically zero. Show that

$$\Gamma^1_{11} = \frac{1}{2}\frac{E_u}{E}\,,$$

and find similar formulae for the other Γ^k_{ij} $(i, j, k = 1, 2)$.

Show that $LN - M^2$ is expressible in terms of these functions and E, G and their derivatives. Deduce that the Gauss curvature K is expressible in terms of E, G and their derivatives. [You need not find the exact expression for $LN - M^2$ or for K.]

8. (a) Let M be a surface. Define what is meant by a *(smooth) Riemannian metric on* M.

(b) Let $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ and let g be the hyperbolic metric on $\mathbb{R}^2_+$ given at a point $p = (x,y)$ of $\mathbb{R}^2_+$ by

$$g_p(\mathbf{v}, \mathbf{w}) = \frac{1}{y^2}\, \mathbf{v} \cdot \mathbf{w}\,.$$

Show that the following bijective smooth maps of $(\mathbb{R}^2_+,\, g)$ are isometries:

$$\begin{array}{rrcll} \text{(i)} & \psi(x,y) & = & (x+\lambda,\, y) & (\lambda \in \mathbb{R})\,, \\ \text{(ii)} & \psi(x,y) & = & (\lambda x,\, \lambda y) & (\lambda \in \mathbb{R},\ \lambda \neq 0)\,, \\ \text{(iii)} & \psi(x,y) & = & (-x,\, y)\,, & \\ \text{(iv)} & \psi(x,y) & = & \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)\,. & \end{array}$$

[You do not need to show that these maps are smooth and bijective, and you may assume that a smooth bijective map $\psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is an isometry if and only if, for each $p \in \mathbb{R}^2_+$, there is a basis $\{e_1, e_2\}$ of $T_p\mathbb{R}^2_+$ such that $g_{\psi(p)}\big(\mathrm{d}\psi_p(e_i), \mathrm{d}\psi_p(e_j)\big) = g_p(e_i, e_j)$ $(i = 1, 2)$.]

(c) Find an isometry of $(\mathbb{R}^2_+, g)$ which preserves the semicircle of centre $(0,0)$, radius 3, but is not the identity map. Hence find $a \in (0, \infty)$ such that the distance from $(0,1)$ to $(0,2)$ is equal to the distance of $(0,a)$ to $(0,9)$ without calculating these distances.

END