MATH–5031M01

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Only approved basic scientific calculators may be used.

Examination for the Module MATH–5031M

(January 2005)

Differential Geometry 2

Time allowed: 3 hours

Answer a maximum of **four** questions from Section A and a maximum of **two** questions from Section B. All questions carry equal marks.

SECTION A

Throughout Section A, by 'surface' we shall mean 'smooth regular embedded m-surface in $\mathbb{R}^n$ for some positive integers m and n'.

1. **(a)** Let $\gamma : [0, b] \to \mathbb{R}^2$ be a smooth unit speed parametrized curve. (i) What is meant by γ is *closed*? (ii) Define the *signed curvature* $\kappa(s)$ of γ at $s \in [0, b]$.

Let $\theta : [0, b] \to \mathbb{R}$ be a smooth function such that $\gamma'(s) = \big(\cos\theta(s), \sin\theta(s)\big) \quad \big(s \in [0, b]\big)$. Show that $\kappa(s) = \theta'(s) \quad \big(s \in [0, b]\big)$.

Hence show that the total curvature $\int_0^b \kappa(s)\,\mathrm{d}s$ of γ is given by $\theta(b) - \theta(0)$, and define the *rotation index* of γ [you need not show that it is an integer].

(b) Let $\gamma(t) = (6\cos 2t, -6\sin 2t) \quad \big(t \in [0, 3\pi]\big)$. Calculate the total curvature of γ and thus its rotation index.

(c) By using part (b) and finding a suitable regular homotopy or otherwise, show that the rotation index of the closed curve $\alpha : [0, 3\pi] \to \mathbb{R}^2$ given by

$$\alpha(t) = \big(6\cos 2t + 2\sin 4t\,,\ -6\sin 2t + 2\cos 4t\big) \qquad \big(t \in [0, 3\pi]\big)$$

is -3.

2. **(a)** Let $f : W \to \mathbb{R}^k$ be a smooth map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^k$ where n and k are positive integers with $k \leq n$. Say what is meant by $p \in W$ is a *regular point*. Let $c \in f(W)$. State a condition on c which ensures that $f^{-1}(c)$ is a (smooth regular embedded) m-surface in $\mathbb{R}^n$ for some m, giving the value of m in terms of n and k.

Hence show that $S_c = \big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = c\big\}$ is a smooth 2-surface if $c \neq 0$.

By calculating the shape operator or otherwise, show that the surface S_1 has Gauss curvature -1 at the point $(1, 0, 0)$.

(b) For any $c \in \mathbb{R}$, let $M_c = \big\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1,\ x_1 + x_3 = c\big\}$. Show that M_c is empty if $|c| > \sqrt{2}$ and is a nonempty (smooth regular embedded) 2-surface in $\mathbb{R}^4$ if $|c| < \sqrt{2}$.

3. **(a)** Let $f : M \to M'$ be a smooth map between surfaces, and let $p \in M$. Define the *differential* $\mathrm{d}f_p : T_pM \to T_{f(p)}M'$ of f at p. Let $g : M' \to M''$ be another smooth map between surfaces and let $p \in M$. Show that $\mathrm{d}(g \circ f)_p = \mathrm{d}g_{f(p)} \circ \mathrm{d}f_p$.

(b) Let $f : M \to M'$ be a smooth map between surfaces. Define what is meant by f is a *local isometry*. Let C be the cylinder $\big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\big\}$. Show that the map $f : \mathbb{R}^2 \to C$ defined by $f(u_1, u_2) = (\cos u_1, \sin u_1, u_2)$ is a local isometry.

Let S be the cone $\big\{(x, y, z) \in \mathbb{R}^3 : a^2x^2 + a^2y^2 = z^2\big\}$ where $a > 0$. Define a map $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to S$ by $f(r\cos\theta, r\sin\theta) = (br\cos 2\theta, br\sin 2\theta, abr)$ where $b > 0$. Show that f is a local isometry if and only if $a = \sqrt{3}$ and $b = 1/2$.

[You may use the fact that a smooth map $f : M \to M'$ is a local isometry if and only if, for each $p \in M$, there is an orthonormal basis $\{\mathbf{e}_i\}$ such that $\{\mathrm{d}f_p(\mathbf{e}_i)\}$ is orthonormal.]

continued . . .

4. **(a)** Let $f : M \to M'$ be a smooth map between 2-surfaces. Say what is meant by f *is conformal with scale factor* λ. Show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if

$$\mathrm{d}f_p(\mathbf{v}) \cdot \mathrm{d}f_p(\mathbf{w}) = \lambda(p)^2\, \mathbf{v} \cdot \mathbf{w} \qquad (p \in M,\ \mathbf{v}, \mathbf{w} \in T_pM)\,.$$

Hence show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if, for all $p \in M$, there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of T_pM such that $|\mathrm{d}f_p(\mathbf{v}_i)| = \lambda(p)\,|\mathbf{v}_i|$ $(i = 1, 2)$ and $\mathrm{d}f_p(\mathbf{v}_1) \cdot \mathrm{d}f_p(\mathbf{v}_2) = \lambda(p)^2\, \mathbf{v}_1 \cdot \mathbf{v}_2$.

(b) Let $\phi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ be defined by

$$\phi(x, y) = \frac{1}{(x^2 + y^2)^k}\,(x, y)$$

where k is a positive constant. Show that ϕ is conformal if and only if $k = 1$. Determine the scale factor of ϕ in this case.

5. **(a)** Let $f : M \to M'$ be a local isometry between surfaces and let $\alpha : I \to M$ be a smooth curve. Show that the length of α is equal to the length of $f \circ \alpha$.

(b) Let M be a 2-surface in $\mathbb{R}^3$. What is meant by a property is (A) *intrinsic*, (B) *extrinsic*. For each of the following properties of M, state which is intrinsic and which is extrinsic, giving a brief reason: (i) principal curvatures; (ii) mean curvature; (iii) length of curves; (iv) Gauss curvature. [You may quote values of the principal curvatures of standard surfaces and the existence of local isometries between some of these surfaces without proof.]

(c) Let M be a closed 2-surface. Explain briefly what is meant by the (i) *total curvature*, (ii) the *Euler characteristic* of M [you need not define what is meant by a triangulation or show that the Euler characteristic is well defined].

State the *Gauss–Bonnet Theorem.*

Use the Gauss–Bonnet Theorem to determine (i) the total curvature of the surface

$$M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + \frac{1}{3}\,y^2 + 2z^2 = 1\right\},$$

(ii) the Euler characteristic of a closed surface whose total curvature is -4π.

SECTION B

6. (a) Let $\alpha : I \to \mathbb{R}^3$ be a unit speed smooth closed curve. Define its *tangential map* $f : I \to S^2$ to the unit sphere S^2 by $f(s) = \alpha'(s)$ $(s \in I)$ and let Γ denote its image. Show that (i) for any $s \in I$, the unsigned curvature $\kappa(s)$ of α at s is equal to the speed of f at s; (ii) the total curvature $\int_I |\kappa(s)| \, \mathrm{d}s$ of α is equal to the length of Γ (i.e., of f).

Now let $\mathbf{a}$ be a fixed unit vector. Define a function $g : I \to \mathbb{R}$ by $g(s) = \mathbf{a} \cdot \alpha(s)$. Show that, if $s \in I$ is a point where g attains a maximum or minimum, then $\mathbf{a} \cdot f(s) = 0$. Deduce that (i) Γ is met by every great circle of S^2, (ii) the length of Γ is at least 2π. Hence show that the total curvature of α is at least 2π.

[You may assume that, if Γ is a smooth closed curve in S^2 of length less than 2π, then there is a point $m \in S^2$ such that the spherical distance of m from x is less than $\pi/2$ for all points $x \in \Gamma$.]

(b) Suppose now that $\alpha : I \to \mathbb{R}^3$ is a nontrivial knot. Can its total curvature be 3π? Explain your answer briefly.

7. Let M be an m-surface in $\mathbb{R}^n$. Let $\alpha : [a, b] \to M$ be a smooth curve on M (which is not necessarily of unit speed or regular). Set

$$E(\alpha) = \frac{1}{2} \int_a^b |\alpha'(t)|^2 \, \mathrm{d}t \, .$$

(i) Find the first variation formula for E.

(ii) Show that, if α is a geodesic, then E is stationary with respect to variations which fix the endpoints.

(iii) Show the converse, i.e., if E is stationary with respect to variations which fix the endpoints, then α is a geodesic. [You may assume that (i) for any closed interval $[a, b]$ and any $t_0 \in [a, b]$, there are a_1, b_1 with $a < a_1 < t_0 < b_1 < b$ and a smooth function $f : [a, b] \to [0, \infty)$ with $f(t_0) > 0$ and $f(t) = 0$ for all $t \notin [a_1, b_1]$; (ii) given a vector field v along α, there is a variation of α with variation vector field v.]

(iv) A variation of α is called *normal* if its variation vector field v satisfies $v(s) \cdot \alpha(s) = 0$ $(s \in I)$ (the variation need not fix the endpoints). How must your proofs in parts (ii) and (iii) be modified to show that α *is a geodesic if and only if* E *is stationary with respect to* normal *variations*?

continued ...

8. **(a)** Let M and M' be 2-surfaces in $\mathbb{R}^3$ and let $f : M \to M'$ be a smooth map. What is meant by saying that f is *equiareal*. Show that f is equiareal if and only if, for each $p \in M$, there is a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of T_pM such that

$$|\mathrm{d}f_p(\mathbf{v}_1) \times \mathrm{d}f_p(\mathbf{v}_2)| = |\mathbf{v}_1 \times \mathbf{v}_2| \,.$$

(b) Define a map f from the unit circular cylinder to the unit sphere by

$$f(\cos t,\, \sin t,\, u) = \left(\sqrt{1-g(u)^2}\, \cos t,\, \sqrt{1-g(u)^2}\, \sin t,\, g(u)\right)$$

where g is a smooth real-valued function with $g(0) = 0$ and $g'(u) > 0$ for all u. Show that f is equiareal if and only if $g(u) = u$.

(c) Show that, if a smooth map between 2-surfaces is both conformal and equiareal, then it is a local isometry.

Hence, show, without any calculation, that stereographic projection is not equiareal.

[You may assume standard properties of conformal maps and that stereographic projection is conformal, and standard consequences of the Theorema Egregium.]

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