MATH445001

This question paper consists of 7 printed pages, each of which is identified by the reference **MATH445001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH4450

(May 2003)

Polymeric Fluids

Time allowed: **3 hours**

Answer FIVE of the SEVEN questions.

All questions carry equal marks.

1. Define the shear viscosity of a fluid in terms of the shear-stress and shear-rate. Explain the meaning of the terms shear-thinning and shear-thickening.

The stress σ in polymeric fluid is related to the shear-rate $\dot{\gamma}$ by

$$\sigma + k^2\sigma^3 = \eta\dot{\gamma},$$

where k and η are positive constants. Show that for small shear stresses, $|\sigma| \ll 1/k$ that the fluid behaves as a Newtonian fluid with viscosity η. Show for $|\sigma| \gg 1/k$ that fluid behaves as a power-law fluid and determine the index of the power-law. Sketch a graph of stress versus shear-rate for this material. Is it shear-thinning or shear-thickening?

This fluid is forced to flow down a channel of width $2h$ by a pressure gradient of magnitude G. Show that the fluid velocity in the channel is given by

$$u = \frac{G}{2\eta}\left(h^2 - y^2\right) + \frac{k^2G^3}{4\eta}\left(h^4 - y^4\right).$$

Find the volume flux per unit length, Q

$$Q = \int_{-h}^{h} u dy,$$

and sketch a curve of $\log Q$ versus $\log G$ indicating the gradient at small and large values of G.

2. A polymeric fluid is contained between two parallel circular disks of radius a that are a distance h apart. The fluid is open to the atmosphere at $r = a$. The upper disk is rotated at angular velocity Ω while the lower disk remains fixed, so that in cylindrical polar coordinates the fluid velocity between the plates is given by

$$\mathbf{u} = \frac{\Omega r z}{h}\hat{\boldsymbol{\theta}}.$$

Find the velocity gradient $\mathbf{K}$ and show that for $r \gg h$ the shear-rate $\dot{\gamma}$ is approximately $\frac{r\Omega}{h}$. Define the normal stress differences N_1 and N_2 in this flow in terms of the components of the extra stress tensor $\boldsymbol{\sigma}$.

Hence show that a normal force equal to

$$F = 2\pi \int_0^a (\tau_{rr} + N_2 + p_{\mathrm{atm}})\, r dr$$

is required to maintain the separation of the plates, where p_{atm} is atmospheric pressure.

Show from the radial momentum equation that

$$\frac{\partial \tau_{rr}}{\partial r} = \frac{N_1 + N_2}{r}.$$

For the case $N_1(\dot{\gamma}) = \Psi_1 \dot{\gamma}^2$ and $N_2(\dot{\gamma}) = \Psi_2 \dot{\gamma}^2$ where Ψ_1 and Ψ_2 are constants, show that

$$\tau_{rr} = -p_{\mathrm{atm}} + \frac{(\Psi_1 + \Psi_2)\Omega^2}{2h^2}\left(r^2 - a^2\right).$$

Hence find the force, F.

3. Write down an expression for the stress in a linear viscoelastic fluid of relaxation modulus $G(t)$. By considering the stress generated by a shear-rate,

$$\dot{\gamma} = \frac{d}{dt}\left(\epsilon \exp(i\omega t)\right),$$

define the complex modulus G^* and explain the significance of the real and imaginary parts of G^*.

The shear stress, $\sigma(t)$ in a linear Maxwell fluid is related to the shear-rate $\dot{\gamma}$ by

$$\tau \frac{d\sigma}{dt} + \sigma = \mu \dot{\gamma}.$$

Show that this is a linear viscoelastic fluid and find its relaxation modulus, $G(t)$.

Find the complex modulus of this fluid and show that the loss and storage moduli are given respectively by

$$G' = \frac{\mu \omega^2 \tau}{1 + \omega^2 \tau^2}, \qquad G'' = \frac{\mu \omega}{1 + \omega^2 \tau^2}.$$

A fluid satisfying the linear Maxwell model is subjected to the following shear flow

$$\dot{\gamma} = \begin{cases} a \sin \omega t & \text{for } t < 0, \\ 0 & \text{for } t \geq 0. \end{cases}$$

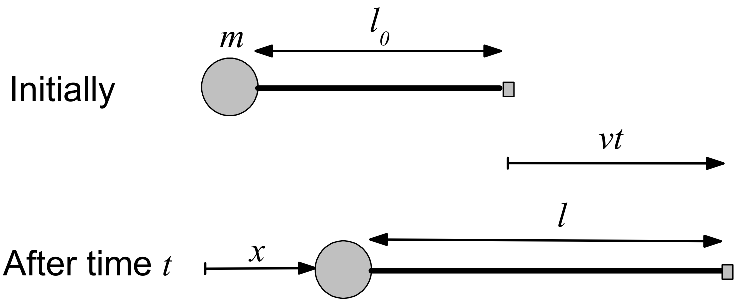
Find the shear stress for $t > 0$.

4. The expression for the total stress in a rubber is

$$\boldsymbol{\tau} = G\mathbf{F}\cdot\mathbf{F}^T - \beta\mathbf{I}.$$

(a) What is the deformation gradient $\mathbf{F}$ and stress $\boldsymbol{\tau}$ for a volume-conserving uniaxial extension by a ratio λ in the x-direction? A piece of rubber, of initial cross sectional area A_0, is stretched by a ratio λ. If the sides of the rubber are exposed to the atmosphere, so that $\tau_{yy} = \tau_{zz} = -p_{atm}$, show that the force required to achieve the stretch is

$$f = GA_0\left(\lambda - \frac{1}{\lambda^2}\right).$$



(b) A mass m is attached to a thin piece of rubber of initial length l_0 and initial cross sectional area A_0, as shown in the diagram. The other end of the rubber is initially held a distance l_0 away (so that the rubber is just taught) and then moved away from the mass at constant velocity, v. Let x be the displacement of the mass from its initial position after time t.

(i) By considering the length, l, of the rubber after time t, obtain expressions for the stretch, λ, of the rubber and the acceleration $\frac{d^2x}{dt^2}$ of the mass (assuming that the only force on the mass is due to the rubber - i.e. it moves without friction)

(ii) Defining $X = x - vt$, show that for small $\frac{X}{l_0}$,

$$\frac{d^2X}{dt^2} = -\frac{3GA_0}{l_0 m}X.$$

(iii) Assuming the mass is initially at rest, obtain a solution for X (and hence, x) as a function of time. For what velocities is the assumption of small $\frac{X}{l_0}$ valid?

(iv) Given that the rubber becomes "slack" for $\lambda < 1$, find the value of X at which this occurs. Show that this occurs first when

$$t = \pi\sqrt{\frac{3GA_0}{l_0 m}}.$$

Briefly describe the subsequent motion of the mass.

5. Two particles, at positions x_1 and x_2, are joined by a spring with spring constant k so that their Langevin equations are

$$\zeta\frac{dx_1}{dt} = k(x_2 - x_1) + f_1(t)$$
$$\zeta\frac{dx_2}{dt} = k(x_1 - x_2) + f_2(t)$$

and $\langle x_\alpha(t) f_\beta(t)\rangle = k_B T\delta_{\alpha\beta}$ (where $\alpha, \beta = 1$ or 2).

(a) Defining the particle separation $r = x_2 - x_1$, show that

$$\zeta\frac{dr}{dt} = -2kr + f_2 - f_1$$

and obtain a similar equation for the centre of mass $R = \frac{x_1+x_2}{2}$.

(b) Defining $Q = \langle r^2\rangle$, show that

$$\frac{dQ}{dt} = -\frac{4k}{\zeta}Q + \frac{4k_B T}{\zeta}$$

and obtain a similar equation for $P = \langle R^2\rangle$.

(c) Solve the equations for P and Q subject to initial conditions $x_1 = x_2 = 0$.
Show that the energy in the spring, $U = \frac{1}{2}kr^2$ approaches an average value of $\frac{1}{2}k_B T$.
What is the diffusion constant for the centre of mass?

6. The Rouse equation for a polymer chain comprising beads with friction constant ζ connected with springs of spring constant k is

$$\zeta\left(\frac{\partial \mathbf{r}_s}{\partial t} - \mathbf{v}(\mathbf{r}_s)\right) = k\frac{\partial^2 \mathbf{r}_s}{\partial s^2} + \mathbf{f}_s,\ s = 0..N$$

This is used to model a chain in a rubber network, by setting

$$\mathbf{r}_s = \mathbf{r}_A \text{ at } s = 0, \text{ and}$$
$$\mathbf{r}_s = \mathbf{r}_B \text{ at } s = N$$

where $\mathbf{r}_A$ and $\mathbf{r}_B$ represent the positions of the crosslink points at the ends of the chain.
We suppose $\mathbf{v}_A = \mathbf{v}(\mathbf{r}_{s=0}) = \frac{d\mathbf{r}_A}{dt}$ and $\mathbf{v}_B = \mathbf{v}(\mathbf{r}_{s=N}) = \frac{d\mathbf{r}_B}{dt}$, and that $\mathbf{v}(\mathbf{r}_s) = \mathbf{v}_A + \mathbf{K}\cdot(\mathbf{r}_s - \mathbf{r}_A)$ where $\mathbf{K}$ is the velocity gradient tensor.

(a) Let $\mathbf{r}_s = \mathbf{r}_A + \frac{s}{N}(\mathbf{r}_B - \mathbf{r}_A) + \mathbf{x}_s$. Show that

$$\frac{\partial \mathbf{r}_s}{\partial t} - \mathbf{v}(\mathbf{r}_s) = \frac{\partial \mathbf{x}_s}{\partial t} - \mathbf{K}\cdot\mathbf{x}_s$$

and that

$$\frac{\partial^2 \mathbf{r}_s}{\partial s^2} = \frac{\partial^2 \mathbf{x}_s}{\partial s^2}$$

Hence, obtain a partial differential equation for the new variable $\mathbf{x}_s$. What are the boundary conditions on $\mathbf{x}_s$ at $s = 0$ and $s = N$?

(b) Ignoring the terms due to velocity gradient $\mathbf{K}$ and random force $\mathbf{f}_s$, show that the relaxation time of the pth normal mode $\mathbf{x}_s = \mathbf{X}_p \sin\left(\frac{\pi p s}{N}\right)$ is

$$\tau_p = \frac{\tau_1}{p^2}$$

where $\tau_1 = \frac{N^2\zeta}{\pi^2 k}$.

(c) Given that this model leads to a time-dependent modulus of form

$$G(t) = G_0 + G_0 \sum_{p=1}^{\infty} \exp\left(-\frac{p^2 t}{\tau_1}\right)$$

obtain approximations of the form $G(t) = ct^{\alpha}$ for $t \gg \tau_1$ and (by approximating the sum as an integral) for $t \ll \tau_1$. Hence sketch a graph of $\log G(t)$ versus $\log t$. Why doesn't $G(t)$ decay to zero?

You may use the result $\int_0^{\infty} dX \exp\left(-X^2\right) = \frac{\sqrt{\pi}}{2}$.

7. The constitutive equation for the Upper Convected Maxwell model is

$$\boldsymbol{\tau} = G\mathbf{A} - \beta\mathbf{I},$$

where the structure tensor $\mathbf{A}$ satisfies

$$\frac{d\mathbf{A}}{dt} = \mathbf{K}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{K}^T - \frac{1}{\tau}\left(\mathbf{A} - \mathbf{I}\right).$$

A fluid that obeys the Upper Convected Maxwell model is subjected to a transient shear flow $\mathbf{u} = (\dot{\gamma} y, 0, 0)$ where

$$\dot{\gamma} = \begin{cases} 0 & \text{for } t < 0, \\ g & \text{for } t \geq 0. \end{cases}$$

Write down the equation for evolution of the tensor, $\mathbf{A}$, for $t \geq 0$ and show that only the components A_{xx}, A_{xy} change with time. Find A_{xy} and A_{xx} as functions of time and show for $0 < t \ll \tau$ that

$$A_{xx} \sim 1 + g^2 t^2.$$

Hence sketch a graph showing the first normal stress difference as a function of time.

Formulae Sheet

Cartesian coordinates

pressure, p, velocity, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$, velocity gradient, $\mathbf{K}$ with $K_{ij} = \frac{\partial u_i}{\partial x_j}$

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \qquad \nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Cylindrical Polar Coordinates

velocity, $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$.

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{\theta r} - \sigma_{r\theta}}{r} \\ \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

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Spherical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi},$$

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial\theta} - \dfrac{v}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u}{\partial\phi} - \dfrac{w}{r} \\ \dfrac{\partial v}{\partial r} & \dfrac{1}{r}\dfrac{\partial v}{\partial\theta} + \dfrac{u}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial v}{\partial\phi} - \dfrac{w}{r}\cot\theta \\ \dfrac{\partial w}{\partial r} & \dfrac{1}{r}\dfrac{\partial w}{\partial\theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial w}{\partial\phi} + \dfrac{u}{r} + \dfrac{v}{r}\cot\theta \end{pmatrix}$$

$$\nabla\cdot\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_{\theta r}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi r}}{\partial\phi} - \frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r} \\ -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\theta}}{\partial\phi} + \frac{\sigma_{\theta r}-\sigma_{r\theta}-\sigma_{\phi\phi}\cot\theta}{r} \\ \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\phi}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_{\theta\phi}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{\sigma_{\phi r}-\sigma_{r\phi}+\sigma_{\phi\theta}\cot\theta}{r} \end{pmatrix}$$

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