MATH445001

This question paper consists of 8 printed pages, each of which is identified by the reference **MATH4450**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH4450
(May/June 2002)

Polymeric Fluids

Time allowed: **3 hours**

Answer FIVE of the SEVEN questions.

All questions carry equal marks.

1. In the power-law fluid model the shear stress, σ is equal to $\sigma = K|\dot{\gamma}|^{n-1}\dot{\gamma}$, where K and n are positive constants.

(a) Explain what is meant by the terms *shear thinning* and *shear thickening* and state the range of values of n for which the power law fluid is shear-thinning or shear-thickening.

(b) A power-law fluid is driven along a cylindrical pipe of radius a by a pressure gradient $\frac{\partial p}{\partial z} = -G$. Show that the shear stress σ is given in terms of the shear-rate $\dot{\gamma}$ by

$$\sigma = -\frac{1}{2}Gr.$$

Find the form of the fluid velocity, w and sketch a graph showing $w(r)$ for (a) $n = 0.5$, (b) $n = 1$ and (c) $n = 2$. Explain the differences in the velocity profiles.

(c) Calculate the volume flux, $Q = 2\pi \int_0^a rwdr$ down the pipe. A Newtonian fluid of viscosity μ and a power-law fluid of index $n = 0.5$ have the same volume flow rate down a pipe of radius a when a pressure gradient G is applied. Find the increase in the volume flow -rate of each fluid if:

(i) the pressure gradient is doubled from G to $2G$,

(ii) the pipe is replaced by a pipe of radius $2a$.

Explain why the volume flow rate of the shear-thinning fluid is now larger in both cases.

2. **(a)** Write down the equations of mass and momentum conservation for an incompressible fluid with pressure, p, density, ρ, extra stress, $\boldsymbol{\sigma}$, and velocity, $\mathbf{u}$, that is subject to a gravitational acceleration $\mathbf{g}$. Under what circumstances can fluid inertia be neglected and how does this simplify these equations?

(b) Using the appropriate formulae for cylindrical polar coordinates write down the velocity gradient for a flow in which the fluid velocity is given in cylindrical polar coordinates (r, θ, z) by $\mathbf{u} = (0, v(r), 0)$. Show that the $r\theta$ component of the strain-rate tensor, $\mathbf{E}$,

$$E_{r\theta} = \frac{1}{2}\dot{\gamma} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{v}{r}\right),$$

where $\dot{\gamma}$ is the local shear-rate. Define the shear viscosity, $\mu(\dot{\gamma})$, and first and second normal stress differences, $N_1(\dot{\gamma})$ and $N_2(\dot{\gamma})$, in terms of the components of the extra stress tensor $\boldsymbol{\sigma}$.

(c) A vertical rod of radius a rotates at an angular velocity Ω in a polymeric fluid in which

$$\mu(\dot{\gamma}) = \mu_0, \qquad N_1(\dot{\gamma}) = A\dot{\gamma}^2, \qquad N_2(\dot{\gamma}) = 0,$$

where μ_0 and A are both positive constants. Write down the components of the momentum equation on the assumption that fluid inertia is negligible, and show that this leads to the following equations

$$\frac{\mu_0}{r^2}\frac{\partial}{\partial r}\left(r^2\dot{\gamma}\right) = 0,$$
$$\frac{\partial}{\partial r}\left(-p + \sigma_{zz}\right) = \frac{A}{r}\dot{\gamma}^2,$$
$$\frac{\partial}{\partial z}\left(-p + \sigma_{zz}\right) = \rho g.$$

Hence find the fluid velocity $v(r)$ and show that

$$\dot{\gamma} = -\frac{2\Omega a^2}{r^2}.$$

(d) If the top surface is open to the atmosphere, show that the position of this surface is given by

$$h(r) = h_\infty + \frac{A\Omega^2 a^2}{\rho g r^4},$$

where h_∞ is the height for $r \to \infty$.

3. The extra stress $\boldsymbol{\sigma}$ in the linear Maxwell model is related to the strain-rate by

$$\tau\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{\sigma} = 2\mu\mathbf{E}(t).$$

Show that this may be written in the form

$$\boldsymbol{\sigma} = 2\int_{-\infty}^{t} G(t - t')\mathbf{E}(t')dt',$$

QUESTION 3 CONTINUED...

for some suitable choice for the relaxation modulus $G(t)$. Show that

$$\int_0^\infty G(t)dt = \mu.$$

Find the shear stress $\sigma_{xy}(t)$ generated by the fluid velocity $\mathbf{u} = (\dot{\gamma}y, 0, 0)$ in the following cases:

(a) $\dot{\gamma} = \begin{cases} k & t < 0, \\ 0 & t \geq 0. \end{cases}$

(b) $\dot{\gamma} = \begin{cases} k & -T \leq t < 0, \\ -k & 0 \leq t \leq T, \\ 0 & |t| > T. \end{cases}$

For each case sketch graphs of σ_{xy} and $\dot{\gamma}$ as functions of time. For case (c) show that there must be a time t_0 in the interval $0 < t_0 < T$ at which $\sigma_{xy} = 0$. Find the value of t_0.

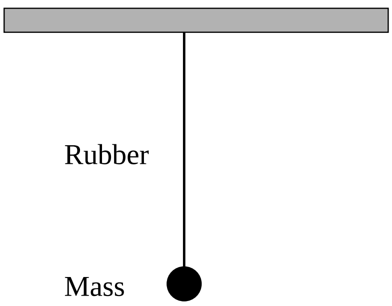
4. The expression for the total stress in a rubber is

$$\boldsymbol{\tau} = G\mathbf{F} \cdot \mathbf{F}^T - \beta \mathbf{I}.$$

(a) What is the deformation gradient, $\mathbf{F}$, and stress, $\boldsymbol{\tau}$, for uniaxial extension by a ratio λ in the z-direction? A piece of rubber, of initial cross sectional area A_0 is stretched by a ratio λ. If the sides of the rubber are exposed to the atmosphere, so that $\tau_{xx} = \tau_{yy} = -p_{atm}$, show that the force required to achieve the stretch is

$$f = GA_0 \left(\lambda - \frac{1}{\lambda^2} \right).$$

(b) A mass m is suspended from a piece of rubber of initial length l_0 and cross sectional area A_0, as shown in the diagram.



(i) What is the relationship between the stretch λ, the length of the rubber, l, and the initial length, l_0?

(ii) By balancing the forces on the mass, obtain a relation between the mass m and the equilibrium stretch of the rubber λ_{eq}.

QUESTION 4 CONTINUED...

(iii) What is the change in λ for a small downward displacement x from the equilibrium position? By considering the forces on the mass after such a displacement, show that

$$m\frac{d^2x}{dt^2} = -\frac{GA_0}{l_0}\left(1+\frac{2}{\lambda_{eq}^3}\right)x.$$

(iv) Hence show that the time period for small vertical oscillations is

$$T_v = 2\pi\left(\frac{l_0}{g}\frac{\lambda_{eq}^4-\lambda_{eq}}{\lambda_{eq}^3+2}\right)^{\frac{1}{2}},$$

where g is the acceleration due to gravity.

5. The Langevin equation for a particle in a quadratic potential $U = \frac{1}{2}kx^2$ is

$$\zeta\frac{dx}{dt} = -kx + f(t),$$

where $\langle f(t) f(t')\rangle = 2k_{\mathrm{B}}T\zeta\delta(t-t')$.

(a) Show that the solution of this equation, subject to initial condition $x(0) = 0$, is

$$x(t) = \frac{1}{\zeta}\int_0^t dt' f(t')\exp\left(\frac{t'-t}{\tau}\right),$$

where $\tau = \frac{\zeta}{k}$.

(b) Hence show that

$$\left\langle x(t)^2\right\rangle = \frac{k_{\mathrm{B}}T}{k}\left(1-\exp\left(-\frac{2t}{\tau}\right)\right).$$

(c)

(i) Obtain the limiting form of $\left\langle x(t)^2\right\rangle$ for $t \ll \tau$ and compare your result with free-particle diffusion (where $\left\langle x(t)^2\right\rangle = 2Dt$).

(ii) Obtain the limiting form of $\left\langle x(t)^2\right\rangle$ for $t \gg \tau$. In this limit, show that the average energy $\langle U\rangle$ approaches $\frac{1}{2}k_{\mathrm{B}}T$.

6. The Rouse equation for a polymer chain comprising beads with friction constant ζ connected with springs of spring constant k is

$$\zeta\left(\frac{\partial \mathbf{r}_s}{\partial t} - \mathbf{v}(\mathbf{r}_s)\right) = k\frac{\partial^2\mathbf{r}_s}{\partial s^2} + \mathbf{f}_s,\ s = 0..N,$$

with boundary conditions

$$\left.\frac{\partial \mathbf{r}_s}{\partial s}\right|_{s=0} = 0 \text{ and } \left.\frac{\partial \mathbf{r}_s}{\partial s}\right|_{s=N} = 0.$$

(a) In terms of the forces acting on a bead, briefly discuss the origin of the term, $\frac{\partial^2\mathbf{r}_s}{\partial s^2}$.

(b) Ignoring the terms due to velocity, $\mathbf{v}(\mathbf{r}_s)$, and random force, $\mathbf{f}_s$, show that the relaxation time of the pth normal mode, $\mathbf{r}_s = \mathbf{X}_p \cos\left(\frac{\pi p s}{N}\right)$, is

$$\tau_p = \frac{\tau_1}{p^2},$$

where $\tau_1 = \frac{N^2\zeta}{\pi^2 k}$.

(c) Given that this leads to a time-dependent modulus of form

$$G(t) = G_0 \sum_{p=1}^{\infty} \exp\left(-\frac{p^2 t}{\tau_1}\right),$$

use $G^* = G' + iG'' = \int_0^\infty i\omega G(s) \exp(-i\omega s)\, ds$ to show that

$$G' = G_0 \sum_{p=1}^{\infty} \frac{\omega^2\tau_1^2}{p^4 + \omega^2\tau_1^2},$$

$$G'' = G_0 \sum_{p=1}^{\infty} \frac{p^2\omega\tau_1}{p^4 + \omega^2\tau_1^2},$$

and obtain approximations of the form $G'' = c\omega^\alpha$ for $\omega\tau_1 \ll 1$ and (by approximating the sum as an integral) for $\omega\tau_1 \gg 1$. Hence sketch a graph of $\log G''$ versus $\log\omega$.

You may use the results:

$$\sum_1^\infty p^{-2} = \frac{\pi^2}{6},$$

$$\int_0^\infty \frac{x^2}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}.$$

7. The stress in the upper convected Maxwell model is given by

$$\boldsymbol{\tau} = -\beta\mathbf{I} + G\mathbf{A},$$

where the second rank tensor $\mathbf{A}$ satisfies

$$\frac{D\mathbf{A}}{Dt} = \mathbf{K}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{K}^{\mathrm{T}} - \frac{1}{\tau}(\mathbf{A} - \mathbf{I}),$$

and $K_{ij} = \frac{\partial u_i}{\partial x_j}$ is the velocity gradient.

Fluid is placed in the gap between two plates of surface area S located at $z = \pm h$. Each plate is coated with a lubricant so that the fluid can slip at the plate surfaces, so that the fluid velocity between the plates is of the form

$$\mathbf{u} = (E(t)x, E(t)y, -2E(t)z).$$

Show that

$$E(t) = -\frac{1}{2h}\frac{dh}{dt}.$$

Find $E(t)$ if the plates are squeezed together so that

$$h(t) = \begin{cases} h_0 \mathrm{e}^{-\frac{2t}{\tau}} & 0 \leq t \leq \tau, \\ h_0 e^{-2} & t > \tau. \end{cases}$$

The fluid is at equilibrium at $t = 0$, so that $\mathbf{A} = \mathbf{I}$. Deduce that for $t > 0$ the only non-zero components of the $\mathbf{A}$ are A_{xx}, A_{yy} and A_{zz} and that $A_{xx} = A_{yy}$. Find $\mathbf{A}(t)$ for $t > 0$. If the edges of the plates are open to the atmosphere so that $\tau_{xx} = \tau_{yy} = -p_{\text{atm}}$, show that the net force exerted by the fluid on the plates,

$$F = GS(A_{xx} - A_{zz})$$

Find $F(t)$ and explain why F is non-zero for $t > \tau$?

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Formulae Sheet

Cartesian coordinates

pressure, p, velocity, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$, velocity gradient, $\mathbf{K}$ with $K_{ij} = \frac{\partial u_i}{\partial x_j}$

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \qquad \nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Cylindrical Polar Coordinates

velocity, $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$.

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{\theta r} - \sigma_{r\theta}}{r} \\ \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

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Spherical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{w}{r} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} - \frac{w}{r}\cot\theta \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{w}{r} + \frac{v}{r}\cot\theta \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta r}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} - \frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r} \\ \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{\sigma_{\theta r}-\sigma_{r\theta}-\sigma_{\phi\phi}\cot\theta}{r} \\ \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\phi}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\phi}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{\phi r}-\sigma_{r\phi}+\sigma_{\phi\theta}\cot\theta}{r} \end{pmatrix}$$

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