MATH353101

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH3531**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH3531
(January 2005)

Cosmology

Time allowed: **2 hours**

Attempt not more than FOUR questions.
All questions carry equal marks.

[You may assume the following:

The special relativistic Lorentz transformations for two inertial frames in standard configuration with relative speed v are

$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - xv/c^2)$, where $\gamma \equiv (1 - v^2/c^2)^{-\frac{1}{2}}$.]

The Friedmann equations, where all terms have their usual meanings, are

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}G\pi\rho - \frac{kc^2}{R^2} + \frac{1}{3}\Lambda, \qquad \frac{\ddot{R}}{R} = -\frac{4}{3}\pi G\left(\rho + 3\frac{p}{c^2}\right) + \frac{1}{3}\Lambda.$$

Dots denote differentiation with respect to time t.]

1. (a) Two bodies A and B have masses M_A and M_B, and the distance between them is a fixed value r_{AB}. Show that the stagnation point S between them at which a particle is equally attracted to both is at a distance r_{AS} from A where

$$r_{AS} = \frac{r_{AB}}{1 + (M_B/M_A)^{\frac{1}{2}}}.$$

Show that this position of equilibrium is unstable. Find the value of $r_{ES}/r_{E\mu}$ for the Earth-Moon (E-μ) system given that $M_E = 5.98\times 10^{24}$kg and $M_\mu = 7.336 \times 10^{22}$kg.

(b) A photon is projected vertically upwards from a point A and moves in a constant gravitational field, whose acceleration is g, to a point B at a height L above A. Using a crude calculation involving only special relativity and quantum mechanics, find a relation between the frequencies ν_A and ν_B at A and B respectively. What happens to the colour of the light in rising from A to B?

2. **(a)** A source of monochromatic light with rest wavelength λ_E and rest frequency ν_E is moving radially with speed v relative to an observer O. If λ_O is the wavelength, and ν_O is the frequency, observed by O, show that these quantities are related by

$$\frac{\lambda_O}{\lambda_E} = \frac{\nu_E}{\nu_O} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}.$$

Show that the corresponding non-relativistic limit is

$$\frac{\lambda_O}{\lambda_E} = \frac{\nu_E}{\nu_O} = 1 + \frac{v}{c}.$$

The Cassini – Huygens spacecraft arrived at the planet Saturn in July 2004. The system will release the probe Huygens to explore Saturn's moon Titan in January 2005, and the data obtained from this probe will be sent to Earth by first relaying it to Cassini. Because of the relative speed between Cassini and Huygens, the frequency ν_O of the signals received at Cassini must differ from the emitting frequency ν_E at Huygens. However, last minute inflight tests have shown that ν_E has been wrongly set to ν'_E, so that data from Huygens would be lost at the proposed relative speed. NASA scientists have decided that data transfer could be accomplished by altering this proposed relative speed. Assuming that all relative motion is radial and that the non-relativistic approximation applies, calculate the change in relative speed necessary to achieve communication.

(b) The frequency ν_{peak} corresponding to the peak of the energy distribution of black-body radiation at temperature T is given approximately by $\nu_{peak} \approx 2.8k_BT/h$ where k_B is Boltzmann's constant and h is Planck's constant. This implies that the quantity ν_{peak}/T is a constant. Evaluate this constant, clearly stating the units. The Sun radiates approximately as a blackbody with $T_{Sun} \approx 5800$K. Evaluate ν_{peak} for solar radiation. [Take $k_B = 1.381 \times 10^{-23}\text{JK}^{-1}$ and $h = 6.626 \times 10^{-34}$Js.]

3. A uniform gas of volume V at pressure p undergoes a reversible expansion.

(i) Show, using a simple example of a gas expanding in a cylinder of constant cross section, that the rate of change of energy is $\dot{E} = -p\dot{V}$.

(ii) The density of material in the universe is ρ and the pressure is p. Using the energy – mass relation from special relativity, derive the fluid equation

$$\dot{\rho} + \frac{3\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0$$

where R is the scale factor of the universe. How does this result differ from its counterpart derived without taking special relativity into account?

(iii) The pressure-density relation for an *ideal quantum gas* is given by $p = g\rho c^2$ where g is a constant. Use the fluid equation to show that $\rho R^{3(1+g)}$ is a constant in time.

(iv) The universe has a scale factor $R = R_1$ and density $\rho = \rho_1$ at some initial time t_1. If the universe is treated as an ideal quantum gas and $(k = 0, \Lambda = 0)$, show that

$$R = \left({R_1}^{\frac{3}{2}(1+g)} \pm A(t - t_1)\right)^{\frac{2}{3(1+g)}}$$

QUESTION 3 CONTINUED...

where

$$A \equiv \frac{3}{2}(1+g)\left(\frac{8}{3}\pi G\rho_1 R_1{}^{3(1+g)}\right)^{\frac{1}{2}}.$$

What is the significance of the $\pm$ sign?

4. (a) Define the *deceleration parameter* q_2. Explain why this definition is a more satisfactory definition than the quantity $-\ddot{R}$. Show that

$$q_2 = \frac{1}{H^2}\left[\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) - \frac{1}{3}\Lambda\right].$$

An ideal quantum gas has a pressure-density relation in the form $p = g\rho c^2$ where g is a positive constant. What is the relationship between Λ and ρ in this case if the rate of expansion of the universe is found to be increasing?

(b) In the case of dust and $\Lambda = 0$, you may *assume* that the Friedmann equations may be written

$$\dot{R}^2 = \frac{Q}{R} - kc^2, \qquad \ddot{R} = -\frac{4\pi G}{3}R\rho_d$$

where ρ_d is the dust density and $Q \equiv (8\pi/3)GR^3\rho_d$ is a constant. Define a new variable $u \equiv \sqrt{c^2R/Q}$. Show that, during the present expansion phase,

$$H = \frac{c^3}{Qu^3}\sqrt{1-ku^2} \quad \text{and} \quad q_2 = \frac{1}{2(1-ku^2)}.$$

Assuming that $R = 0$ at time $t = 0$, show that the age of the universe is

$$t = \frac{Q}{c^3}I_k(u)$$

where

$$I_k(u) \equiv 2\int_0^u \frac{x^2}{\sqrt{1-kx^2}}dx.$$

Evaluate t in terms of the Hubble time $t_H \equiv 1/H$ for the case $k = 0$.

5. (a) You may *assume* for the case of zero pressure that the Friedmann equations can be written

$$\ddot{R} = -\frac{Q}{2R^2} + \frac{1}{3}\Lambda R, \qquad \dot{R}^2 = \frac{Q}{R} - kc^2 + \frac{1}{3}\Lambda R^2,$$

where Q is a positive constant. On defining

$$R^* \equiv \frac{3Q}{2c^2}, \qquad \Lambda^* \equiv \frac{4c^6}{9Q^2},$$

show that the Friedmann equations can be written in the form

$$\ddot{x} = \frac{\beta}{x^2}(\alpha x^3 - 1), \qquad \dot{x}^2 = \frac{\beta}{x}(2 + \alpha x^3 - 3kx),$$

where $x \equiv R/R^*$, $\alpha \equiv \Lambda/\Lambda^*$ and $\beta \equiv \Lambda^*/3$.

(b) Obtain expressions for H and q_2 as functions of x. Use these results to show that

(i) In models with $\Lambda < 0$ the scale factor R reaches a maximum value, and that $q_2 > 0$;

(ii) In Einstein's model of a static universe, $R = R^*$ and $\Lambda = \Lambda^*$.

END