MATH 343301

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH 3433**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH 3433
(January 2006)

Electromagnetism

Time allowed: **3 hours**

Do not answer more than FOUR questions. All questions carry equal weight.

The following vector identities may be helpful.
For vector fields $\mathbf{a}$ and $\mathbf{b}$ and a scalar field f:

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (f\mathbf{a}) = f(\nabla \cdot \mathbf{a}) + \mathbf{a} \cdot \nabla f$$

$$\nabla \times (f\mathbf{a}) = \nabla f \times \mathbf{a} + f \nabla \times \mathbf{a}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b})$$

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1. A spatially bounded charge distribution occupying a volume V has charge density ρ and electric scalar potential ϕ. Show that the electrostatic energy W can be written as

$$W = \frac{1}{2} \int_V \rho \phi dV. \tag{1}$$

Show, using the equations of electrostatics, that this may be written as

$$W = \frac{\epsilon_0}{2} \int \mathbf{E}^2 dV, \tag{2}$$

where the integral is taken over all space, and where ϵ_0 is the permittivity.

Total charge Q is distributed with charge density $\rho = \rho_0 r/a$ throughout the sphere $r < a$. Explain why the electric field takes the form $\mathbf{E} = (E(r), 0, 0)$ in spherical polar coordinates. Use Gauss's law to calculate $E(r)$ in terms of Q for $r < a$ and $r > a$. Hence calculate the potential ϕ. Using expression (1) show that

$$W = \frac{Q^2}{7\pi\epsilon_0 a}.$$

Now check your answer by using expression (2).

2. State clearly Gauss's law in integral form, and derive from it the law in differential form.

Consider an infinite straight line charge, with uniform density q per unit length, lying along the z-axis. In which direction is the electric field? Show that the electric potential of this charge distribution is, in cylindrical polar coordinates,

$$\phi = \frac{-q}{2\pi\epsilon_0}\ln r,$$

where r represents the distance from the z-axis.

Now suppose that an infinite line charge, uniform density $+q$ per unit length, lies along the line $x = d/2, y = 0, -\infty < z < \infty$ and that a parallel line charge, of density $-q$ per unit length, lies along the line $x = -d/2, y = 0, -\infty < z < \infty$. Using the above result for a single line charge write down the potential for this system of two line charges. Denote the vector separation of the two line charges by $\mathbf{d}$. A *line dipole* is formed by letting $q \to \infty$, $\mathbf{d} \to 0$ in such a way that $\mathbf{p} = q\mathbf{d}$ remains finite. Show that, in cylindrical polars with origin situated on the dipole axis, the electric potential is given by

$$\phi = \frac{\mathbf{p}\cdot\mathbf{r}}{2\pi\epsilon_0 r^2}.$$

Show that the equipotentials are cylindrical surfaces. Calculate the electric field.

3. What is the physical significance of the equation

$$\nabla\cdot\mathbf{B} = 0, \tag{1}$$

where $\mathbf{B}$ is the magnetic field? Explain how the result (1) implies the existence of a vector potential $\mathbf{A}$. Explain why the vector potential is not uniquely defined.

Write down the equation governing the magnetic field and the current density $\mathbf{j}$ for the case of steady current flow. Show that, under a certain condition on $\mathbf{A}$, which should be stated, $\mathbf{A}$ and $\mathbf{j}$ are related by the equation

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{j}. \tag{2}$$

Suppose that a steady current flows in a bounded conductor occupying a volume V. By comparison of (2), in Cartesian coordinates, with a similar equation that arises in electrostatics, show that the vector potential $\mathbf{A}$ at a point P with position vector $\mathbf{r}$, lying outside V, is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}d^3\mathbf{r}'\,.$$

Hence show that the magnetic field $\mathbf{B}$ at P is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{(\mathbf{r}'-\mathbf{r})\times\mathbf{j}}{|\mathbf{r}'-\mathbf{r}|^3}d^3\mathbf{r}'\,. \tag{3}$$

If the current flows only in a closed wire loop C, with line element $\mathbf{ds}$, and has constant magnitude I, show that equation (3) becomes

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{(\mathbf{r}' - \mathbf{r}) \times \mathbf{ds}}{|\mathbf{r}' - \mathbf{r}|^3}. \tag{4}$$

A steady current I flows round an elliptical wire loop located in the plane $z = 0$ and described by the equation $x^2/a^2 + y^2/b^2 = 1$. Using (4), show that the magnetic field at the point with coordinates $(0, 0, z)$ is

$$\mathbf{B} = \frac{\mu_0 I ab}{4\pi} \int_0^{2\pi} \frac{d\theta}{\left(a^2 \cos^2\theta + b^2 \sin^2\theta + z^2\right)^{3/2}} \hat{\mathbf{z}}.$$

4. Consider time-harmonic electromagnetic fields of the form

$$\mathbf{E} = \mathrm{Re}(\mathbf{E}' \exp(-i\omega t)), \qquad \mathbf{H} = \mathrm{Re}(\mathbf{H}' \exp(-i\omega t)),$$

where $\mathbf{E}'$ and $\mathbf{H}'$ are constant complex vectors. The flux of electromagnetic energy is given by the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. Show that the time-averaged Poynting vector can be expressed as

$$\overline{\mathbf{S}} = \frac{1}{2}\mathrm{Re}(\mathbf{E}' \times \mathbf{H}'^*),$$

where * denotes the complex conjugate.

Now consider a plane electromagnetic wave propagating in the positive z-direction through free space (with permittivity ϵ_0 and permeability μ_0), with $\mathbf{E}$ and $\mathbf{H}$ given by

$$\mathbf{E} = \mathrm{Re}(\mathbf{E}_0 \exp i(kz - \omega t)), \qquad \mathbf{H} = \mathrm{Re}(\mathbf{H}_0 \exp i(kz - \omega t)), \tag{$\dagger$}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are constant complex vectors. State the relation between k, ω, ϵ_0 and μ_0. Use Maxwell's equations to show that $\overline{\mathbf{S}}$ can be written as

$$\overline{\mathbf{S}} = \left(\frac{1}{2c\mu_0}\right) |\mathbf{E}_0|^2 \hat{\mathbf{z}},$$

where c is the speed of light.

A wave of form ($\dagger$) is incident from free space in $z < 0$ onto the plane surface $z = 0$. The half-space $z > 0$ is filled with dielectric of permittivity ϵ_1 and permeability μ_0. The electric field in $z < 0$ may be expressed as the superposition of incident and reflected waves as follows:

$$\mathbf{E} = \mathbf{E}_i \exp i(kz - \omega t) + \mathbf{E}_r \exp i(-kz - \omega t).$$

The transmitted electric field in $z > 0$ takes the form:

$$\mathbf{E} = \mathbf{E}_t \exp i(k_t z - \omega t).$$

 QUESTION 4 CONTINUED...

Use Maxwell's equations to calculate the magnetic fields in the two regions. By applying appropriate boundary conditions at $z = 0$ obtain expressions for $\mathbf{E}_r$ and $\mathbf{E}_t$ in terms of $\mathbf{E}_i$. Show that R, the ratio of (time-averaged) reflected energy flux to incident flux, and T, the ratio of (time-averaged) transmitted flux to incident flux, are given by

$$R = \left|\frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}}\right|^2, \quad T = \frac{4\sqrt{\epsilon_r}}{\left|1+\sqrt{\epsilon_r}\right|^2},$$

where $\epsilon_r = \epsilon_1/\epsilon_0$. What is the physical significance of the result $R + T = 1$?

5. State Maxwell's equations governing electromagnetic fields in free space. From these, show that in free space the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ satisfy the wave equation, namely

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}, \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B},$$

where c is the speed of light.

Consider the rectangular waveguide consisting of a hollow cavity enclosed by perfectly conducting walls at $x = 0$, $x = a$, $y = 0$, $y = b$. What boundary conditions must $\mathbf{E}$ and $\mathbf{B}$ satisfy on the walls? Consider the electric field

$$\mathbf{E} = \Big[\alpha \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \beta \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right),$$
$$\gamma \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\Big] \exp i(kz - \omega t),$$

with m and n integers, not both zero. Show that $\mathbf{E}$ satisfies the appropriate boundary conditions. Use the equation for $\nabla \cdot \mathbf{E}$ in the absence of charge to express γ in terms of α and β. Use Faraday's law to obtain $\mathbf{B}$. Check that the equation $\nabla \cdot \mathbf{B} = 0$ is then satisfied identically. Use any component of the remaining Maxwell equation, in the absence of current, to obtain the dispersion relation

$$\frac{\omega^2}{c^2} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + k^2.$$

Suppose $a > b$. Find the cut-off frequency below which waves cannot propagate. Find the range of ω such that only one mode can propagate. How does this depend on the ratio a/b?

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