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Examination for the Module MATH-3224

(January 2007)

Topology

Time allowed: 3 hours

Attempt no more than **four** questions. All questions carry equal marks.

1. (a) Define the following mathematical terms:

   (i). A *topology* τ on a set X.

   (ii). A *Hausdorff* topological space.

   (iii). A *closed* subset of a topological space.

   (iv). The *closure* $\overline{A}$ of a subset A of a topological space.

   (v). The *interior* A° of a subset A of a topological space.

   (vi). The *boundary* ∂A of a subset A of a topological space.

   (vii). A *continuous* map $f : X \to Y$ between topological spaces X and Y.

   (b) Let $\sigma = \{U \subset \mathbb{R} \,:\, U \text{ is finite}\} \cup \{\emptyset, \mathbb{R}\}$. Show that σ is *not* a topology on $\mathbb{R}$.

   (c) Let $\tau = \{U \subset \mathbb{R} \,:\, \text{if } 0 \in U \text{ then } U = \mathbb{R}\}$.

   (i). Show that τ is a topology on $\mathbb{R}$ and determine, clearly explaining your reasoning, whether $(\mathbb{R}, \tau)$ is Hausdorff.

   (ii). Let $A = (-1, 1)$ and $B = (0, 1)$. Write down the closure, interior and boundary of each of these sets in $(\mathbb{R}, \tau)$.

   (iii). Let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ be continuous with $f(0) \neq 0$. Prove that f is constant.

   (iv). Let $g : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ be any function with $g(0) = 0$. Prove that g is continuous.

   (v). Let τ_* be the usual topology on $\mathbb{R}$ and $h : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_*)$ such that $h(x) = x^3$. Is h continuous? Carefully explain your answer.

□

2. (a) Let (X, τ) be a topological space and A be a subset of X.

(i). Define the *subspace topology* τ_A on A.

(ii). Show that the inclusion map $\iota : (A, \tau_A) \to (X, \tau)$, $\iota(x) = x$, is continuous.

(iii). Let $X = \mathbb{R}$, τ be the usual topology on $\mathbb{R}$ and $A = [0, 1] \cup \mathbb{Z}$. Determine, clearly explaining your reasoning, whether the following sets are open in (A, τ_A):

$$B = \{0, 1\}, \qquad C = \{2, 3\}, \qquad D = [0, 1).$$

(b) (i). Define the terms *connected topological space* and *connected subset* of a topological space.

(ii). Let X be a connected topological space, Y be a topological space and $f : X \to Y$ be a continuous map. Prove that $f(X)$ is a connected subset of Y.

(iii). Let $\{A_\lambda \,:\, \lambda \in \Lambda\}$ be an indexed family of connected subsets of a topological space X with the property that $\bigcap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Prove that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is a connected subset of X.

(iv). Using the above results, prove that the unit 2-sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 + x_3^2 = 1\}$$

is a connected subset of $\mathbb{R}^3$ (with the usual topology). You may assume the following facts without proof (where $\mathbb{R}$, $\mathbb{R}^3$ have the usual topology):

- Every interval is a connected subset of $\mathbb{R}$.
- A function $f : Z \to \mathbb{R}^3$, $f(z) = (f_1(z), f_2(z), f_3(z))$, is continuous if each $f_i : Z \to \mathbb{R}$, $i = 1, 2, 3$, is continuous

□

3. (a) Define the following mathematical terms:

(i). An *open cover* of a topological space.

(ii). A *compact* topological space.

(iii). A *compact subset* of a topological space.

(b) Let σ_1, σ_2 be the following collections of subsets of $\mathbb{R}$:

$$\begin{aligned} \sigma_1 &= \{(a, \infty) \,:\, a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}, \\ \sigma_2 &= \{U \subset \mathbb{R} \,:\, \mathbb{R} \backslash U \text{ is finite}\} \cup \{\emptyset\}. \end{aligned}$$

You are given that σ_1, σ_2 are topologies on $\mathbb{R}$.

(i). Show that $(\mathbb{R}, \sigma_1)$ is noncompact.

(ii). Show that $(\mathbb{R}, \sigma_2)$ is compact.

(c) Let X be a compact topological space, Y be a topological space and $f : X \to Y$ be continuous. Prove that $f(X)$ is a compact subset of Y.

Continued ...

(d) Let X be a Hausdorff topological space and A be a compact subset of X. Prove that A is closed.

(e) Let τ_* be the usual topology on $\mathbb{R}$ and τ be any compact topology on $\mathbb{R}$. Let

$$f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_*) \quad \text{such that} \quad f(x) = \frac{1}{1+x^2}.$$

Using the results of parts (c) and (d), or otherwise, show that f is not continuous. □

4. (a) Define the following mathematical terms:

(i). A *metric* d on a set X.

(ii). A *Cauchy* sequence in (X, d).

(iii). A *complete* metric space.

(iv). A *contraction mapping*.

(v). A *fixed point* of a mapping.

(b) (i). Let $\varphi : X \to X$ be a contraction mapping. Prove that φ is sequentially continuous.

(ii). Let $\varphi : X \to X$ be a contraction mapping, $x_1 \in X$ and the sequence (x_n) be defined by $x_n = \varphi(x_{n-1})$ for all $n \geq 2$. Prove that (x_n) is Cauchy.

(iii). State and prove the Contraction Mapping Theorem.

(c) Use the Contraction Mapping Theorem to prove that the equation

$$x^4 - 2x^2 + 16x + 7 = 0$$

has one and only one solution in $[-1, 1]$.

□

5. Determine whether each of the following statements is true or false. If the statement is true, prove it. If the statement is false, give a counterexample, explaining why it is a counterexample.

(a) Let X be a Hausdorff space and $f : X \to Y$ be continuous and surjective. Then Y is Hausdorff.

(b) For all subsets A, B of a topological space X, $\overline{A} \cap \overline{B} = \overline{A \cap B}$.

(c) Let A be a compact subset of a topological space X. Then A is closed.

(d) Let $f : X \to Y$ be continuous. Then f is sequentially continuous.

(e) Let A be a closed and bounded subset of a metric space (X, d). Then A is compact.

(f) Let A be a compact subset of a metric space (X, d). Then $X \backslash A$ is noncompact.

(g) Let X be a topological space, Y be a Hausdorff space, $f : X \to Y$ and $g : X \to Y$ be continuous, and

$$C = \{x \in X \ : \ f(x) = g(x)\}.$$

Then C is closed.

$\square$

End.