MATH–322401

Only approved basic scientific calculators may be used.

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–3224

Examination for the Module MATH–3224

(January 2001)

TOPOLOGY

Time allowed : 3 hours

Do not answer more than four questions. All questions carry equal marks.

1. What axioms does the collection τ of open sets in a topological space (X, τ) satisfy?

Define a collection σ of subsets of the set $\mathbb{Z}$ of integers by specifying that σ consist of $\emptyset$, $\mathbb{Z}$, and all subsets W_n, $n \geq 0$, where $W_n = \{k \in \mathbb{Z} : |k| \leq n\}$. Show that σ satisfies the axioms for a topology on $\mathbb{Z}$.

What is meant by saying that a subset S of a topological space (X, τ) is *closed?* Give examples of subsets of $(\mathbb{Z}, \sigma)$ that are (i) neither open nor closed; (ii) both open and closed.

Prove that if $\{F_\lambda : \lambda \in \Lambda\}$ is a family of closed sets in (X, τ) then $\bigcap_{\lambda \in \Lambda} F_\lambda$ is closed.

Define the *closure*, $\overline{S}$, the *interior*, $\operatorname{int} S$, and the *boundary*, ∂S, of a subset S of a topological space.

Show from your definition that the boundary of a set is always closed.

List the closed sets of $(\mathbb{Z}, \sigma)$, and hence for each of the following subsets of $(\mathbb{Z}, \sigma)$ calculate its closure, interior and boundary.

(i) $A = \{-1, 0, 1\}$; (ii) $B = \{2\}$; (iii) $C = \{k \in \mathbb{Z} : k \geq 0\}$.

2. (a) State the axioms for a metric space (X, d).

Let (X, d) be a metric space. What is meant by saying that a sequence (x_n) in X is a *Cauchy sequence*?

What is meant by saying that a metric space is *complete*?

Show that every closed subset of a complete metric space is complete.

(b) For vectors $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$ in $\mathbb{R}^2$ we define

$$d(\mathbf{v}_1, \mathbf{v}_2) = |x_1 - x_2| + |y_1 - y_2|.$$

Show that $(\mathbb{R}^2, d)$ is a metric space.

Let $(\mathbf{v}_n) = (x_n, y_n)$ be a Cauchy sequence in $(\mathbb{R}^2, d)$. Assuming without proof that $\mathbb{R}$ is complete in its usual metric, show that there exist real numbers x and y such that $x_n \to x$ and $y_n \to y$, and deduce that $(\mathbb{R}^2, d)$ is complete.

(c) What does it mean to say that T is a *contraction mapping* on a metric space (X, d)?

State the Contraction Mapping Theorem, and prove the uniqueness part of that theorem.

Fix $\mathbf{z} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$. Suppose that T is a contraction mapping on $(\mathbb{R}^2, d)$. Show that the mapping $U : \mathbf{x} \mapsto \mathbf{z} - \lambda T\mathbf{x}$ is a contraction mapping and deduce that the equation $\mathbf{x} + \lambda T\mathbf{x} = \mathbf{z}$ has a unique solution for $\mathbf{x}$.

3. (a) What is meant by saying that a topological space (X, τ) is *connected*?

Define the *subspace topology* on a subset S of a topological space (X, τ). What does it mean to say that S is connected?

Write down, without proof, a result which states precisely which subsets of the real line (with the standard topology) are connected.

Prove directly that $\mathbb{Q}$ is disconnected.

What does it mean to say that a topological space is *path-connected*?

(b) Describe briefly the construction of the Cantor set C. Prove that C is closed and disconnected.

(c) Let (X, σ) and (Y, τ) be topological spaces. State carefully what is meant by saying that a mapping $f : X \to Y$ is (i) *continuous*; (ii) a *homeomorphism*.

Show that (in the Euclidean topology) no two of the following three sets are homeomorphic (standard results on connectedness may be used without proof).

(i) The Cantor set C; (ii) The unit interval $[0, 1]$; (iii) The unit square $[0, 1] \times [0, 1]$.

State precisely, but without proof, the Peano–Hilbert theorem on the existence of a space-filling curve.

4. (a) What is meant by saying that a topological space (X, τ) is *Hausdorff*?

What does it mean to say that a sequence (x_n) in a topological space (X, τ) *converges* to a point x in X?

Question 4 continues . . .

Show that, if X is Hausdorff and (x_n) is a sequence in X such that $x_n \to x$ and $x_n \to y$ for some points x, $y \in X$, then $x = y$.

Give an example of a topological space that has a sequence in it converging to more than one limit.

(b) What is meant by saying that a topological space (X, τ) is *normal*?

Show that every metric space is normal (you may assume without proof that, for every closed subset of A, the function $x \mapsto \text{dist}(x, A)$ is continuous, provided that you define it correctly).

(c) Let $\mathbb{R}$ have the topology τ in which the open sets are $\emptyset$, $\mathbb{R}$ and sets of the form $U_a = \{x \in \mathbb{R} : x > a\}$ for $a \in \mathbb{R}$.

What are the closed subsets of $\mathbb{R}$ in this topology?

Show that $(\mathbb{R}, \tau)$ is normal but not Hausdorff.

5. **(a)** What is meant by saying that a subset K of a topological space is *compact*?

Show that if K and L are compact, then $K \cup L$ is compact.

What is meant by saying that a subset K of a metric space (X, d) is *precompact (totally bounded)*? Show that every compact subset of a metric space is precompact.

(b) Let (X, σ) and (Y, τ) be topological spaces. Define the *product topology* $\sigma \times \tau$ on $X \times Y$.

Suppose that $X \times Y$ is compact in the product topology. Prove that X and Y are both compact.

(c) What is meant by saying that $(A, \leq)$ is a *directed set*?

Let (X, τ) be a topological space, let $(A, \leq)$ be a directed set, let $\phi : A \to X$ be a net, and let $y \in X$. What does it mean to say that ϕ *converges to* y?

Let S be a subset of a topological space. Show that if $y \in \overline{S}$ then there is a net ϕ in S which converges to y.

END