MATH–321401

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Examination for the Module MATH–3214

(January 2005)

Fourier analysis

Time allowed : 3 hours

Answer not more than **four** questions. All questions carry equal marks.

All functions in this paper are assumed to be Riemann integrable on any finite interval.

1. (a) What is meant by saying that a function $f : \mathbb{R} \to \mathbb{C}$ has *period* T? Give an example of a (non-constant) function that has period 2.

(b) Define the *complex Fourier coefficients* $\hat{f}(n)$, $n \in \mathbb{Z}$, of a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$.

For $k \in \mathbb{Z}$, let $e_k(x) = e^{ikx}$. Show that $\widehat{e_k}(n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$

Let p be the trigonometric polynomial given by $p(x) = \sum_{k=-N}^{N} c_k e_k(x)$. Show that

$$\hat{p}(n) = \begin{cases} c_n & \text{if } |n| \leqslant N, \\ 0 & \text{otherwise.} \end{cases}$$

The function g is defined by $g(x) = \sin^2 x$. Express g as a linear combination of the functions e_k, and hence find its Fourier coefficients $\hat{g}(n)$.

(c) Define the *Fourier cosine* and *sine coefficients* a_n, b_n of a 2π-periodic function f, and obtain formulas for a_n, b_n in terms of $\hat{f}(n)$ and $\hat{f}(-n)$.

2. (a) What does it mean to say that a 2π-periodic function f has an *absolutely convergent Fourier series*?

Let f be the 2π-periodic function defined on $(-\pi, \pi]$ by $f(x) = x^2$. Show that the Fourier coefficients of f are given by $\hat{f}(n) = 2(-1)^n/n^2$ for $n \neq 0$, and find $\hat{f}(0)$.

Explain why the Fourier series of f converges to f. By making use of this convergence at the point $x = \pi$, prove that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

(b) Define the *convolution* $f * g$ of two continuous 2π-periodic functions f, $g : \mathbb{R} \to \mathbb{C}$.

For a fixed integer p, let $g(x) = e^{ipx}$ and let $h = f * g$. Show that $h = \hat{f}(p)g$.

3. Define the *inner-product norm* $\|f\|$ of a continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$. Prove that $\|f - s_k(f)\|^2 = \|f\|^2 - \sum_{n=-k}^{k} |\hat{f}(n)|^2$, where $s_k(f)(x) = \sum_{n=-k}^{k} \hat{f}(n)e^{inx}$ and k is a positive integer. Hence show that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$.

By calculating the Fourier coefficients of the 2π-periodic function f defined on $(-\pi, \pi]$ by $f(x) = e^{\lambda x}$ (where $\lambda > 0$), show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{\lambda^2 + n^2} \leqslant \frac{\pi \cosh \lambda\pi}{\lambda \sinh \lambda\pi}.$$

4. **(a)** Define the *Fourier transform* $\hat{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying $\int_{\mathbb{R}} |f(x)|\,dx < \infty$.

For $a \in \mathbb{R}$ and $b > 0$, define functions $T_a f$ and $D_b f$ by

$$(T_a f)(x) = f(x - a), \qquad (D_b f)(x) = f(x/b).$$

Obtain formulas for the Fourier transforms $\widehat{T_a f}(w)$ and $\widehat{D_b f}(w)$.

(b) Let g denote the Gaussian function $g(x) = e^{-x^2}$, for $x \in \mathbb{R}$. Assuming without proof that it is legitimate to differentiate the integral defining $\hat{g}$, show that

$$\hat{g}'(w) = -\tfrac{1}{2} w \hat{g}(w), \quad \text{for } w \in \mathbb{R}.$$

Given that $\int_{\mathbb{R}} g(x)\,dx = \sqrt{\pi}$, calculate $\hat{g}(w)$ for $w \in \mathbb{R}$.

Using the first part of the question, find the Fourier transform of the function $f(x) = e^{-x^2/2}$.

5. **(a)** Define the *inverse Fourier transform* $\check{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ with $\int_{\mathbb{R}} |f(w)|\,dw < \infty$.

Find $\check{f}$ when the function f is defined by

$$f(w) = \begin{cases} 1 & \text{when } |w| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let $K_1 : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

(*i*) $K_1(x) \geqslant 0$ for all $x \in \mathbb{R}$,

(*ii*) $\int_{\mathbb{R}} K_1(x)\,dx = 1$

(*iii*) $K_1(x) \leqslant C/(1 + x^2)$, for some constant $C > 0$.

Also, let K_m be defined by $K_m(x) = mK_1(mx)$, for each $m \in \mathbb{N}$. Show that, for a continuous function $f : \mathbb{R} \to \mathbb{C}$ satisfying $\int_{\mathbb{R}} |f(x)|\,dx < \infty$,

$$(K_m * f)(x) \to f(x) \quad \text{as } m \to \infty,$$

for each $x \in \mathbb{R}$.

END