MATH–321401

Only approved basic scientific calculators may be used.

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–3214

Examination for the Module MATH–3214

(January 2003)

FOURIER ANALYSIS

Time allowed : 3 hours

Do not answer more than **four** questions. All questions carry equal marks.

All functions in this paper are assumed to be Riemann integrable on any finite interval.

1. **(a)** Define the *complex Fourier coefficients* $\hat{f}(n)$, $n \in \mathbb{Z}$, of a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$.

Let $e_k(x) = e^{ikx}$, for $k \in \mathbb{Z}$. Show that

$$\hat{e}_k(n) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the Fourier coefficients of the functions $\cos kx$ and $\sin kx$ for $k = 1, 2, \ldots$

Let the function $p : \mathbb{R} \to \mathbb{C}$ have the form $p(x) = \sum_{k=1}^{m} a_k \cos kx + b_k \sin kx$ for some $m \in \mathbb{N}$ and real numbers a_k and b_k. Calculate $\hat{p}(n)$ for $n \in \mathbb{Z}$.

(b) You are given that $\sqrt{2}$ is an irrational number. By considering real solutions to the equation $f(x) = 2$, or otherwise, show that the function f defined by $f(x) = \cos x + \cos(x\sqrt{2})$ is not periodic.

(c) The 2π-periodic function f is defined on $(-\pi, \pi]$ by $f(x) = |x|$.

Show that

$$\int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2((-1)^n - 1)}{n^2}$$

for $n \in \mathbb{Z} \setminus \{0\}$, and hence calculate the Fourier coefficients of f.

Assuming without proof that the Fourier series for f converges to $f(0)$ at the point $x = 0$, deduce that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

2. **(a)** What does it mean to say that a 2π-periodic function f has an *absolutely convergent Fourier series*?

State without proof a sufficient condition (in terms of the smoothness of f) for f to have an absolutely convergent Fourier series.

Determine, giving brief reasons, which of the following 2π-periodic functions defined by their values on $(-\pi, \pi]$ have absolutely convergent Fourier series. (There is no need to calculate any Fourier coefficients explicitly.)

(i) $f(x) = \cos^4 3x$; (ii) $f(x) = e^{3x}$; (iii) $f(x) = 1/(2 - \cos x)$.

(b) The Rudin–Shapiro trigonometric polynomials P_n and Q_n for $n = 0, 1, 2, \dots$ are defined inductively by the formulae $P_0 \equiv 1$, $Q_0 \equiv 1$, and

$$P_{n+1}(x) = P_n(x) + e^{i2^n x} Q_n(x),$$
$$Q_{n+1}(x) = P_n(x) - e^{i2^n x} Q_n(x),$$

for $n \geq 0$ and $x \in \mathbb{R}$.

Prove that $|P_n(x)|^2 + |Q_n(x)|^2 = 2^{n+1}$, and deduce that $|P_n(x)| \leq 2^{(n+1)/2}$ for all $n \geq 0$ and $x \in \mathbb{R}$.

Explain briefly why the function f, defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} e^{i2^n x} P_n(x) \qquad \text{for} \quad x \in \mathbb{R},$$

is continuous and 2π-periodic on $\mathbb{R}$, and show that it does not have an absolutely convergent Fourier series.

3. **(a)** Define the *convolution* $f * g$ of two continuous 2π-periodic functions f, $g : \mathbb{R} \to \mathbb{C}$.

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous 2π-periodic function and let $p(x) = \displaystyle\sum_{k=-m}^{m} c_k e^{ikx}$ be a trigonometric polynomial.

Derive a formula for $f * p(x)$ in terms of the constants c_k and the Fourier coefficients $\hat{f}(k)$.

(b) Define the *Dirichlet kernel functions* $(D_m)_{m=0}^{\infty}$, and give the Fourier coefficients $\hat{D}_m(k)$ for $k \in \mathbb{Z}$.

Show that, if f is a 2π-periodic function, then $(f * D_m)(x) = \sum_{k=-m}^{m} \hat{f}(k) e^{ikx}$.

Deduce that for m, $n \in \mathbb{N}$, we have $D_m * D'_n = D'_p$, where $p = \min(m, n)$ and the notation D'_n denotes the derivative of D_n.

(c) State Fejér's theorem on the summability of the Fourier series of continuous 2π-periodic functions.

continued ...

4. **(a)** For f, $g : \mathbb{R} \to \mathbb{C}$ continuous and 2π-periodic we write

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx,$$

and, for $k \in \mathbb{Z}$, e_k denotes the function defined on $\mathbb{R}$ by $e_k(x) = e^{ikx}$.

Prove that, for any positive integer k,

$$\left\langle f - \sum_{n=-k}^{k} \langle f, e_n \rangle e_n,\ f - \sum_{n=-k}^{k} \langle f, e_n \rangle e_n \right\rangle = \langle f, f \rangle - \sum_{n=-k}^{k} |\langle f, e_n \rangle|^2.$$

Deduce Bessel's inequality (which should be stated precisely).

(b) Let f be continuous and 2π-periodic, and suppose that its derivative f' is also continuous. Prove that $(f')\hat{}(n) = in\hat{f}(n)$ for $n \in \mathbb{Z}$.

State Parseval's identity and use it to prove that if f is 2π-periodic with a continuous derivative, and $\hat{f}(0) = 0$, then

$$\int_{-\pi}^{\pi} |f'(x)|^2\, dx \geq \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Give an example to show that the inequality can fail to hold if $\hat{f}(0) \neq 0$.

5. **(a)** Define the *Fourier transform* $\hat{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty. \qquad (*)$$

Let g denote the function defined by $\quad g(x) = \begin{cases} 1 - x^2 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$

Show that

$$\hat{g}(w) = \int_{-1}^{1} (1 - x^2) \cos wx\, dx,$$

and hence derive the expression $\quad \hat{g}(w) = \dfrac{4(\sin w - w \cos w)}{w^3} \quad$ for $\quad w \neq 0$.

Define the *inverse Fourier transform* $\check{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying (*).

Give a precise statement of Fourier's Inversion Theorem. Assuming without proof that the conditions of the theorem apply to the function g defined above, calculate

$$\int_{-\infty}^{\infty} \hat{g}(w) \cos wx\, dw \qquad \text{for} \quad x \in \mathbb{R}.$$

(b) Give a precise definition of the *Schwartz class* $\mathcal{S}$ of smooth, rapidly-decreasing functions, and state without proof a theorem describing the Fourier transforms of functions in $\mathcal{S}$.

END