MATH–321401

Only approved basic scientific calculators may be used.

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–3214

Examination for the Module MATH–3214

(January 2002)

FOURIER ANALYSIS

Time allowed : 3 hours

Do not answer more than **four** questions. All questions carry equal marks.

All functions in this paper are assumed to be Riemann integrable on any finite interval.

1. (a) Define the *complex Fourier coefficients* $\hat{f}(n)$, $n \in \mathbb{Z}$, of a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$.

Let $e_k(x) = e^{ikx}$, for $k \in \mathbb{Z}$. Show that

$$\hat{e}_k(n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Let the function $p : \mathbb{R} \to \mathbb{C}$ have the form $p = \sum_{k=-m}^{m} c_k e_k$ for some $m \in \mathbb{N}$ and complex numbers c_k. Show that

$$\hat{p}(n) = \begin{cases} c_n & \text{if } |n| \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

The function h is defined by $h(x) = \cos^2 x$. Express h in terms of the functions e_k, and hence calculate its Fourier coefficients $\hat{h}(n)$.

(b) The 2π-periodic function f is defined on $(-\pi, \pi]$ by $f(x) = e^x$. What is $f(20\pi)$?

Show that

$$\hat{f}(n) = \frac{(-1)^n \sinh \pi}{\pi(1 - in)}, \qquad \text{for } n \in \mathbb{Z}.$$

Assuming without proof that the Fourier series for f converges to $f(0)$ at the point $x = 0$, deduce that

$$\sum_{n \in \mathbb{Z}} \frac{(-1)^n}{1 + n^2} = \frac{\pi}{\sinh \pi}.$$

2. **(a)** What does it mean to say that a 2π-periodic function f has an *absolutely convergent Fourier series*?

Show that if f has an absolutely convergent Fourier series, then its Fourier series converges uniformly to a continuous function g such that $\hat{g}(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$.

State without proof a sufficient condition (in terms of the smoothness of f) for f to have an absolutely convergent Fourier series.

(b) Let f be continuous and 2π-periodic, and suppose that its derivative f' is also continuous. Prove that $(f')\hat{\ }(n) = in\hat{f}(n)$ for $n \in \mathbb{Z}$.

For a fixed r with $0 < r < 1$ the *Poisson kernel function* $P_r : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and infinitely differentiable, and its Fourier coefficients satisfy $\hat{P}_r(n) = r^{|n|}$ for $n \in \mathbb{Z}$.

Calculate the Fourier coefficients of its kth derivative $P_r^{(k)}$ for $k \in \mathbb{N}$, and deduce that each $P_r^{(k)}$ has an absolutely convergent Fourier series.

(c) Give (without proof) an example of a continuous 2π-periodic nowhere-differentiable function, expressible as the sum of a Fourier series.

Explain briefly why the formula you have given defines a continuous function.

3. **(a)** Define the *convolution* $f * g$ of two continuous 2π-periodic functions f, $g : \mathbb{R} \to \mathbb{C}$.

Show that $f * e_k = \hat{f}(k)e_k$, where e_k denotes the function $e_k(x) = e^{ikx}$ for $k \in \mathbb{Z}$.

Give without proof a formula for $(f * g)\hat{\ }(n)$ in terms of $\hat{f}(n)$ and $\hat{g}(n)$, for $n \in \mathbb{Z}$.

Define the *Dirichlet kernel functions* $(D_k)_{k=0}^{\infty}$ in terms of the functions e_k, and hence give an expression for $f * D_k$ in terms of the Fourier coefficients of f. Show also that

$$D_k(x) = \frac{\sin\left(k + \frac{1}{2}\right)x}{\sin\frac{x}{2}},$$

whenever x is not a multiple of 2π. What value does D_k take at multiples of 2π?

(b) Let $(K_m)_{m=0}^{\infty}$ denote the Fejér kernel functions given by

$$K_m = \frac{1}{m+1}\sum_{k=0}^{m} D_k.$$

Calculate the Fourier coefficients $\hat{K}_m(n)$ for $n \in \mathbb{Z}$.

State Fejér's theorem on the summability of Fourier series, and use it to deduce a corollary about the approximation of continuous functions by trigonometric polynomials.

4. For $f,\, g : \mathbb{R} \to \mathbb{C}$ continuous and 2π-periodic we write

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx,$$

and, for $k \in \mathbb{Z}$, e_k denotes the function defined on $\mathbb{R}$ by $e_k(x) = e^{ikx}$.

Prove that, for any positive integer k,

$$\left\langle f - \sum_{n=-k}^{k} \langle f, e_n\rangle e_n,\; f - \sum_{n=-k}^{k} \langle f, e_n\rangle e_n \right\rangle = \langle f, f\rangle - \sum_{n=-k}^{k} |\langle f, e_n\rangle|^2. \qquad (*)$$

State without proof a theorem concerning the mean-square convergence of Fourier series of continuous 2π-periodic functions, and use it, together with (*) above, to deduce Parseval's identity.

Show that

$$\int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2((-1)^n - 1)}{n^2}$$

for $n \in \mathbb{Z} \setminus \{0\}$, and hence calculate the Fourier coefficients of the function f defined on $[-\pi, \pi]$ by $f(x) = |x|$.

Use Parseval's identity to deduce that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$

5. (a) Define the *Fourier transform* $\hat{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty. \qquad (**)$$

Let g denote the Gaussian function $g(x) = e^{-x^2}$, for $x \in \mathbb{R}$. Assuming without proof that it is legitimate to differentiate the integral defining $\hat{g}$, show that

$$\hat{g}'(w) = -\frac{w}{2}\hat{g}(w), \qquad \text{for } w \in \mathbb{R}.$$

Given that $\int_{-\infty}^{\infty} g(x)\, dx = \sqrt{\pi}$, calculate $\hat{g}(w)$ for $w \in \mathbb{R}$.

State the Riemann–Lebesgue Lemma for Fourier transforms and use it to deduce that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(w) = \dfrac{w^2 \sin w}{w^2 + 1}$ is not the Fourier transform of any function f satisfying (**).

(b) Define the *inverse Fourier transform* $\check{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying (**).

For $n \in \mathbb{Z}$ find $\check{f}_n$, where f_n is the function defined by

$$f_n(w) = \begin{cases} e^{inw} & \text{if } |w| \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Give a precise statement of Fourier's Inversion Theorem.

END