MATH–311201

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–311201

Only approved basic scientific calculators may be used.

Examination for the Module MATH–3112

(January 2007)

Differential Geometry 1

Time allowed: 2 hours

Do not answer more than **four** questions.
All questions carry equal marks.

Throughout this paper, by 'surface' we shall mean 'smooth regular embedded m-surface in $\mathbb{R}^n$ for some positive integers m and n'.

1. (a) Let $\gamma : [0, b] \to \mathbb{R}^2$ be a smooth unit speed parametrized curve and let $s \in [0, b]$. What is meant by (i) the *unit positive tangent* of γ at s, (ii) the *unit positive normal* of γ at s, the *signed curvature* $\kappa(s)$ of γ at s?

Let $\theta : [0, b] \to \mathbb{R}$ be a smooth function such that $\gamma'(s) = \big(\cos\theta(s), \sin\theta(s)\big) \quad \big(s \in [0, b]\big)$. Show that $\kappa(s) = \theta'(s) \quad \big(s \in [0, b]\big)$.

Now assume that γ is closed. Show that the total curvature $\int_0^b \kappa(s)\,\mathrm{d}s$ of γ is given by $\theta(b) - \theta(0)$, and define the *rotation index* of γ (you need not show that it is an integer). What is meant by the rotation index of a smooth closed regular parametrized curve which is not of unit speed?

(b) Let $\gamma(t) = (r \sin bt, r \cos bt) \quad \big(t \in [0, 2\pi]\big)$, where r and b are positive real constants. Calculate the rotation index of γ.

(c) By using part (b) and finding a suitable regular homotopy, or otherwise, find the rotation index of the closed curve $\alpha : [0, 2\pi] \to \mathbb{R}^2$ given by

$$\alpha(t) = \big(4\sin 3t - \sin t\,,\; 4\cos 3t - \cos t\big) \qquad \big(t \in [0, 2\pi]\big)\,.$$

2. **(a)** Let n and k be positive integers with $k \leq n$ and let $f : W \to \mathbb{R}^k$ be a smooth map from an open subset W of $\mathbb{R}^n$ to $\mathbb{R}^k$. Say what is meant by (i) $p \in W$ is a *regular point* of f, (ii) $p \in W$ is a *critical point* of f, (iii) $c \in f(W)$ is a *regular value* of f.

Define a smooth mapping $f : \mathbb{R}^4 \to \mathbb{R}^2$ by $f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2,\, 2x_3 - x_4)$. Find the critical points of f and show that $(1, 0)$ is a regular value. Deduce that

$$M = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 = 1,\; x_4 = 2x_3\}$$

is a smooth regular embedded 2-surface in $\mathbb{R}^4$.

(b) Let $X : \mathbb{R}^2 \to \mathbb{R}^4$ be the smooth map defined by $X(u_1, u_2) = (\cos u_1\,,\, \sin u_1\,,\, u_2\,,\, 2u_2)$. Show that X is regular and has image the 2-surface M above. Show that X is not injective on $\mathbb{R}^2$. Find a maximal open subset of $\mathbb{R}^2$ on which it is injective.

3. **(a)** Let $f : M \to M'$ be a smooth map between surfaces, and let $p \in M$. Define the *differential* $\mathrm{d}f_p : T_pM \to T_{f(p)}M'$ of f at p. Let $g : M' \to M''$ be another smooth map between surfaces and let $p \in M$. Show that $\mathrm{d}(g \circ f)_p = \mathrm{d}g_{f(p)} \circ \mathrm{d}f_p$.

(b) Let $f : M \to M'$ be a smooth map between surfaces. Define what is meant by f is a (i) *local isometry*, (ii) *global isometry.*

What is meant by the *distance* $d(p, q)$ between two points p and q on a surface M?

Define regular maps $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ by $\phi(u, v) = (\sinh v \sin u, -\sinh v \cos u, u)$ and $\psi(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$. Let H denote the image of ϕ and C the image of ψ. Define a map $f : H \to C$ by

$$f\big(\phi(u, v)\big) = \psi(u, v) \qquad \big((u, v) \in \mathbb{R}^2\big).$$

Show that this is (i) well defined, (ii) a local isometry, (iii) not a global isometry. Show that there are points $p, q \in H$ such that $d\big(f(p), f(q)\big) \neq d(p, q)$.

[You may use the fact that a smooth map $f : M \to M'$ is a local isometry if and only if, for each $p \in M$, there is a basis $\{\mathbf{e}_i\}$ such that $\mathrm{d}f_p(\mathbf{e}_i) \cdot \mathrm{d}f_p(\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j \;\; (i, j = 1, 2)$.]

continued ...

4. **(a)** Let $f : M \to M'$ be a smooth map between 2-surfaces. Say what is meant by f *is conformal with scale factor* λ. Show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if

$$\mathrm{d}f_p(\mathbf{v}) \cdot \mathrm{d}f_p(\mathbf{w}) = \lambda(p)^2\, \mathbf{v} \cdot \mathbf{w} \qquad (p \in M,\ \mathbf{v}, \mathbf{w} \in T_pM)\,.$$

Hence show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if, for all $p \in M$, there exists a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of T_pM such that $|\mathrm{d}f_p(\mathbf{e}_i)| = \lambda(p)\,|\mathbf{e}_i|\ \ (i = 1, 2)$ and $\mathrm{d}f_p(\mathbf{e}_1) \cdot \mathrm{d}f_p(\mathbf{e}_2) = \lambda(p)^2\, \mathbf{e}_1 \cdot \mathbf{e}_2$.

(b) Let $S^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$. Define a smooth map $f : \mathbb{R}^2 \to S^2$ by

$$f(x, y) = \frac{1}{1 + x^2 + y^2}\,(2x\,,\ 2y\,,\ 1 - x^2 - y^2) \qquad \big((x, y) \in \mathbb{R}^2\big).$$

Show that ϕ is conformal and determine its scale factor $\lambda : \mathbb{R}^2 \to (0, \infty)$.

Show that f does not map any non-constant parametrized geodesic in $\mathbb{R}^2$ to a parametrized geodesic on S^2.

5. **(a)** Let M be a 2-surface in $\mathbb{R}^3$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis for the tangent space at a point p of M. Write $E = \mathbf{e}_1 \cdot \mathbf{e}_1\,,\ F = \mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_1\,,\ G = \mathbf{e}_2 \cdot \mathbf{e}_2\,,\ L = S(\mathbf{e}_1) \cdot \mathbf{e}_1\,,\ M = S(\mathbf{e}_1) \cdot \mathbf{e}_2 = S(\mathbf{e}_2) \cdot \mathbf{e}_1\,,\ N = S(\mathbf{e}_2) \cdot \mathbf{e}_2$, where S is the shape operator of M at p. Show that the Gauss curvature $K(p)$ of M at p is given by

$$K(p) = \frac{LN - M^2}{EG - F^2}\,.$$

(b) Let M be a 2-surface in $\mathbb{R}^3$. What is meant by a property is (A) *intrinsic*, (B) *extrinsic*. State *two* extrinsic properties of a surface. (You need not justify your answer.)

State the *Theorema Egregium* of Gauss.

(c) Let M be a closed 2-surface in $\mathbb{R}^3$. Explain briefly what is meant by the (i) *total curvature*, (ii) the *Euler characteristic* of M. (You need not define what is meant by a triangulation or show that the Euler characteristic is well defined.)

State the *Gauss–Bonnet Theorem.*

Use the Gauss–Bonnet Theorem to show that the Gauss curvature of a closed orientable 2-surface of genus 2 cannot be identically zero.

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