MATH–311201

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Examination for the Module MATH–3112

(January 2005)

Differential Geometry 1

Time allowed: 2 hours

Do not answer more than **four** questions.
All questions carry equal marks.

Throughout this paper, by 'surface' we shall mean 'smooth regular embedded m-surface in $\mathbb{R}^n$ for some positive integers m and n'.

1. (a) Let $\gamma : [0, b] \to \mathbb{R}^2$ be a smooth unit speed parametrized curve. (i) What is meant by γ is *closed*? (ii) Define the *signed curvature* $\kappa(s)$ of γ at $s \in [0, b]$.

Let $\theta : [0, b] \to \mathbb{R}$ be a smooth function such that $\gamma'(s) = \big(\cos\theta(s), \sin\theta(s)\big) \quad \big(s \in [0, b]\big)$. Show that $\kappa(s) = \theta'(s) \quad \big(s \in [0, b]\big)$.

Hence show that the total curvature $\int_0^b \kappa(s)\,\mathrm{d}s$ of γ is given by $\theta(b) - \theta(0)$, and define the *rotation index* of γ [you need not show that it is an integer].

(b) Let $\gamma(t) = (6\cos 2t, -6\sin 2t) \quad \big(t \in [0, 3\pi]\big)$. Calculate the total curvature of γ and thus its rotation index.

(c) By using part (b) and finding a suitable regular homotopy or otherwise, show that the rotation index of the closed curve $\alpha : [0, 3\pi] \to \mathbb{R}^2$ given by

$$\alpha(t) = \big(6\cos 2t + 2\sin 4t\,,\ -6\sin 2t + 2\cos 4t\big) \qquad \big(t \in [0, 3\pi]\big)$$

is -3.

2. **(a)** Let $f : W \to \mathbb{R}^k$ be a smooth map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^k$ where n and k are positive integers with $k \leq n$. Say what is meant by $p \in W$ is a *regular point*. Let $c \in f(W)$. State a condition on c which ensures that $f^{-1}(c)$ is a (smooth regular embedded) m-surface in $\mathbb{R}^n$ for some m, giving the value of m in terms of n and k.

Hence show that $S_c = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = c\}$ is a smooth 2-surface if $c \neq 0$.

By calculating the shape operator or otherwise, show that the surface S_1 has Gauss curvature -1 at the point $(1, 0, 0)$.

(b) For any $c \in \mathbb{R}$, let $M_c = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1,\ x_1 + x_3 = c\}$. Show that M_c is empty if $|c| > \sqrt{2}$ and is a nonempty (smooth regular embedded) 2-surface in $\mathbb{R}^4$ if $|c| < \sqrt{2}$.

3. **(a)** Let $f : M \to M'$ be a smooth map between surfaces, and let $p \in M$. Define the *differential* $\mathrm{d}f_p : T_pM \to T_{f(p)}M'$ of f at p. Let $g : M' \to M''$ be another smooth map between surfaces and let $p \in M$. Show that $\mathrm{d}(g \circ f)_p = \mathrm{d}g_{f(p)} \circ \mathrm{d}f_p$.

(b) Let $f : M \to M'$ be a smooth map between surfaces. Define what is meant by f is a *local isometry*. Let C be the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. Show that the map $f : \mathbb{R}^2 \to C$ defined by $f(u_1, u_2) = (\cos u_1, \sin u_1, u_2)$ is a local isometry.

Let S be the cone $\{(x, y, z) \in \mathbb{R}^3 : a^2x^2 + a^2y^2 = z^2\}$ where $a > 0$. Define a map $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to S$ by $f(r\cos\theta, r\sin\theta) = (br\cos 2\theta, br\sin 2\theta, abr)$ where $b > 0$. Show that f is a local isometry if and only if $a = \sqrt{3}$ and $b = 1/2$.

[You may use the fact that a smooth map $f : M \to M'$ is a local isometry if and only if, for each $p \in M$, there is an orthonormal basis $\{\mathbf{e}_i\}$ such that $\{\mathrm{d}f_p(\mathbf{e}_i)\}$ is orthonormal.]

4. **(a)** Let $f : M \to M'$ be a smooth map between 2-surfaces. Say what is meant by f *is conformal with scale factor* λ. Show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if

$$\mathrm{d}f_p(\mathbf{v}) \cdot \mathrm{d}f_p(\mathbf{w}) = \lambda(p)^2\, \mathbf{v} \cdot \mathbf{w} \qquad (p \in M,\ \mathbf{v}, \mathbf{w} \in T_pM)\,.$$

Hence show that a smooth map $f : M \to M'$ is conformal with scale factor λ if and only if, for all $p \in M$, there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of T_pM such that $|\mathrm{d}f_p(\mathbf{v}_i)| = \lambda(p)\,|\mathbf{v}_i|$ $(i = 1, 2)$ and $\mathrm{d}f_p(\mathbf{v}_1) \cdot \mathrm{d}f_p(\mathbf{v}_2) = \lambda(p)^2\, \mathbf{v}_1 \cdot \mathbf{v}_2$.

(b) Let $\phi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ be defined by

$$\phi(x, y) = \frac{1}{(x^2 + y^2)^k}\,(x, y)$$

where k is a positive constant. Show that ϕ is conformal if and only if $k = 1$. Determine the scale factor of ϕ in this case.

5. **(a)** Let $f : M \to M'$ be a local isometry between surfaces and let $\alpha : I \to M$ be a smooth curve. Show that the length of α is equal to the length of $f \circ \alpha$.

(b) Let M be a 2-surface in $\mathbb{R}^3$. What is meant by a property is (A) *intrinsic*, (B) *extrinsic*. For each of the following properties of M, state which is intrinsic and which is extrinsic, giving a brief reason: (i) principal curvatures; (ii) mean curvature; (iii) length of curves; (iv) Gauss curvature. [You may quote values of the principal curvatures of standard surfaces and the existence of local isometries between some of these surfaces without proof.]

(c) Let M be a closed 2-surface. Explain briefly what is meant by the (i) *total curvature*, (ii) the *Euler characteristic* of M [you need not define what is meant by a triangulation or show that the Euler characteristic is well defined].

State the *Gauss–Bonnet Theorem.*

Use the Gauss–Bonnet Theorem to determine (i) the total curvature of the surface

$$M = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + \frac{1}{3}\,y^2 + 2z^2 = 1\right\},$$

(ii) the Euler characteristic of a closed surface whose total curvature is -4π.

END