MATH275001

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Examination for the Module MATH2750

(May/June 2004)

INTRODUCTION TO MARKOV PROCESSES

Time allowed: **2 hours**

Attempt no more than FOUR questions.

All questions carry equal marks.

1. (a) Two players A and B gamble with each other, their initial capital being a and b pounds, respectively. Consecutive bets are independent, and at each bet a player wins or loses one pound with equal probability. The game terminates once either of the players has become bankrupt.

Explain how such a game can be modelled with a Markov chain and describe its state space and transition probabilities.

(b) Consider a simple symmetric random walk (i.e., with $p = q = \frac{1}{2}$) on integers between the barriers 0 and 20. Suppose that 20 is an absorbing barrier, while 0 is a reflecting barrier so that the walk currently at 0 will jump to 1 with probability one.

(i) If the walk starts at point i, denote by R_i the probability that the process will never be absorbed. Derive the difference equation

$$R_{i+1} - 2R_i + R_{i-1} = 0 \qquad (1 \le i \le 19).$$

What is the form of the general solution to this equation?

(ii) Explain the boundary condition $R_{20} = 0$ and obtain the boundary condition at 0. Solve the equation and find the function R_i $(0 \le i \le 20)$. Conclude whether the random walk can continue indefinitely without being absorbed.

(c) A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel, then he returns to the airport with probability $\frac{3}{4}$ or goes to the other hotel with probability $\frac{1}{4}$.

Determine the transition matrix for this chain. Assuming that the driver begins at the airport, find the probability of each of his three possible locations after two trips and the probability that he is at hotel B after three trips.

2. (a) Describe what is meant by the terms *transient* and *persistent* as applied to the states of a Markov chain. Give the criterion to discriminate between these cases. Interpret this result in terms of the mean number of visits to the state concerned.

(b) If state i communicates with a transient state j, show that i is transient as well.

(c) Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Draw a transition graph of this chain. Determine which states are transient and which are persistent, and identify all closed irreducible subsets of states. Determine the period of state 6.

3. (a) During one minute, a bacterium splits into two with probability $\frac{1}{4}$, dies with probability $\frac{1}{6}$ or remains intact with probability $\frac{7}{12}$.

(i) Assuming that the population dynamics may be modelled as a discrete-time branching process, obtain the probability of ultimate extinction if there is initially one bacterium.

(ii) Obtain the same probability if the process starts with two bacteria.

(b) Consider a Markov chain (X_n) with state space $S = \{0, 1, 2, \dots\}$ and transition probabilities $p_{i,i+1} = p,\ p_{i0} = q\ \ (i = 0, 1, 2, \dots)$, where $0 < p < 1,\ q = 1 - p$.

(i) Assume that $X_0 = 0$, and let $f_{00}^{(n)}$ be the probability that the first return to the origin occurs at the nth step ($n = 1, 2, \dots$). Show that $f_{00}^{(n)} = p^{n-1}q$ and obtain the generating function

$$F_{00}(s) = \sum_{n=1}^{\infty} f_{00}^{(n)} s^n.$$

(ii) Let f_{00} be the probability of ever returning to state 0, starting from 0. Using $F_{00}(s)$ or otherwise, obtain f_{00} and find the mean recurrence time μ_{00}.

(iii) Consider the n-step transition probabilities $p_{00}^{(n)}$ (e.g., $p_{00}^{(0)} = 1,\ p_{00}^{(1)} = q$), and let $G_{00}(s)$ be the corresponding generating function:

$$G_{00}(s) = \sum_{n=0}^{\infty} p_{00}^{(n)} s^n.$$

Using the Markov property, explain briefly the equation

$$p_{00}^{(n)} = \sum_{i=1}^{n} f_{00}^{(i)} p_{00}^{(n-i)}, \qquad n = 1, 2, \dots,$$

and derive the relation

$$G_{00}(s) - 1 = F_{00}(s)\, G_{00}(s).$$

(iv) Find the function $G_{00}(s)$ using parts (i) and (iii), and expand it as a power series in s to obtain $p_{00}^{(n)}$ for all $n = 0, 1, 2, \dots$

4. (a) A particle performs a random walk on the vertices of a square $A_1A_2A_3A_4$ (with A_2 and A_4 adjacent to A_1). At each step the particle remains where it is with probability $\frac{1}{3}$, and moves to each of its neighbouring vertices with probability $\frac{1}{3}$. If the walk starts at vertex A_1, find

(i) the probability that it will not visit A_4 before its first return to A_1;

(ii) the mean number of steps until it first returns to A_1.

QUESTION 4 CONTINUED...

(b) At the end of a month, a large retail store classifies each of its customer's accounts according to current (0), 30-90 days overdue (1), or more than 90 days overdue (2). Their experience indicates that the accounts move from state to state according to a Markov chain with transition matrix

$$P = \begin{pmatrix} 0.85 & 0.15 & 0 \\ 0.70 & 0.10 & 0.20 \\ 0.15 & 0 & 0.85 \end{pmatrix}$$

(i) Obtain the stationary distribution of the chain.

(ii) In the long run, what fraction of the accounts are in each category? Explain your answer carefully, referring to appropriate theory.

5. (a) Suppose that calls arrive at an answering service according to a Poisson process at the rate of four calls per hour.

(i) What is the probability that fewer than two calls come in the first hour?

(ii) Suppose that six calls arrive in the first hour. What is the probability that at least two calls will arrive in the second hour?

(iii) An operator answering the phones waits until 15 calls have arrived before going to lunch. What is the expected amount of time that the operator will wait?

Obtain the required probabilities to 3 significant figures.

(b) Suppose that in a single server queue, arrivals are discouraged by the queue size on arrival. Specifically, if there are k customers in the system, then the arrival rate is given by $\lambda_k = \lambda/(k+1)$ ($k = 0, 1, 2 \dots$). Service times are assumed to be independent and exponentially distributed with mean $1/\mu$. Let $p_k(t)$ denote the probability that there are k customers in the system at time t.

(i) By considering possible transitions in the system on a small time interval $[t, t+h]$, show that

$$p_0(t+h) = p_0(t)(1 - \lambda h) + p_1(t)\mu h + o(h).$$

Obtain the analogous representation for $p_k(t+h),\ k \geq 1$.

(ii) Passing to the limit as $h \to 0$ derive a set of differential equations for the functions $p_k(t),\ k = 0, 1, 2, \dots$

(iii) Find the stationary distribution $\pi = (\pi_k)$ and verify that the probability that the system is empty is given by $\pi_0 = e^{-\rho}$, where $\rho = \lambda/\mu$.

(iv) What is the mean number in the system and the mean number waiting to be served?

END