MATH261001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH2610**.

Only approved basic scientific calculators may be used

Examination for the Module MATH2610
(January 2007)

Oscillations and Waves

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. **(a)** Write down the equation (Euler-Lagrange equation) satisfied by the function $y(x)$, which takes the prescribed values $y(x_1) = a$, $y(x_2) = b$ and which makes the integral

$$I = \int_{x_1}^{x_2} f(x, y, p) dx$$

stationary, where $p \equiv \frac{dy}{dx}$.

Show that, if f does not depend explicitly on x, then

$$p\frac{\partial f}{\partial p} - f \quad \text{is a constant.}$$

(b) Show that the function $y(x)$ which makes the integral

$$I = \int_0^2 \frac{\sqrt{1+p^2}}{y}\, dx$$

stationary, subject to $y(0) = 1$, $y(2) = 3$, satisfies the equation

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{C}{y}$$

for some constant C.

Solve this equation, subject to the boundary conditions, to find the function $y(x)$.

2. A bead A of mass M slides along a smooth wire inclined at a fixed angle α upwards to the horizontal. Another bead B, also of mass M, hangs from A by a light inextensible string of length ℓ. The system is described by the generalized co-ordinates s and θ, where s is the distance A has moved along the wire and θ is the angle that the string makes with the downward vertical.

With reference to Cartesian co-ordinates x, y, where x is horizontal and y vertical, show that the positions of A and B are, respectively,

$$(s\cos\alpha, s\sin\alpha) \qquad \text{and} \qquad (s\cos\alpha + \ell\sin\theta, s\sin\alpha - \ell\cos\theta).$$

Hence, determine the kinetic energy T, the potential energy V and the Lagrangian L for this system.
Show that Lagrange's equations give

$$2\ddot{s} + \ell\cos(\theta-\alpha)\ddot{\theta} - \ell\sin(\theta-\alpha)\dot{\theta}^2 + 2g\sin\alpha = 0,$$

$$\cos(\theta-\alpha)\ddot{s} + \ell\ddot{\theta} + g\sin\theta = 0.$$

3. A mechanical system, performing small oscillations about a position of stable equilibrium $x = y = 0$, has the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\left(4\dot{x}^2 + 2\dot{x}\dot{y} + \dot{y}^2\right) - \frac{1}{2}\left(4x^2 + y^2\right)$$

where x and y are generalized co-ordinates. Show that the normal frequencies of this system are given by $\omega_1^2 = 2$, $\omega_2^2 = \frac{2}{3}$.

If, at $t = 0$,

$$x = 2 \quad y = -2, \quad \dot{x} = 0, \quad \dot{y} = 0$$

show that the subsequent motion is given by

$$x(t) = \frac{1}{2}\left(3\cos\sqrt{2}\,t + \cos\sqrt{\frac{2}{3}}\,t\right), \qquad y(t) = -3\cos\sqrt{2}\,t + \cos\sqrt{\frac{2}{3}}\,t.$$

4. Show that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on $0 < x < \ell$, $t > 0$, subject to the boundary conditions

$$u = 0, \quad \text{at } x = 0 \text{ and } x = \ell \quad (t > 0),$$

 QUESTION 4 CONTINUED...

can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi ct}{\ell}\right) + D_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right).$$

Find the constants C_n and D_n in the case when initially $\dfrac{\partial u}{\partial t} = 0$ and

$$u = \begin{cases} a_0\, x & (0 < x < \frac{\ell}{4}), \\ \frac{a_0}{3}(\ell - x) & (\frac{\ell}{4} < x < \ell), \end{cases}$$

where a_0 is a constant.

5. **(a)** The small, transverse displacement $u = u(x,t)$ of a stretched string of length ℓ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on $0 < x < \ell$, $t > 0$, subject to the boundary conditions

$$u = 0 \quad \text{at } x = 0 \text{ and } x = \ell \quad (t > 0)$$

and initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t} = g(x) \quad \text{at } t = 0 \quad (0 < x < \ell).$$

The kinetic energy T and the potential energy V of the vibrating string are given by

$$T = \frac{1}{2}\rho_0 \int_0^{\ell} \left(\frac{\partial u}{\partial t}\right)^2 dx \qquad \text{and} \qquad V = \frac{1}{2} T_0 \int_0^{\ell} \left(\frac{\partial u}{\partial x}\right)^2 dx,$$

where ρ_0 and T_0 are, respectively, the equilibrium line density and tension of the string. Show that the total energy $E = T + V$ remains constant.

Use this result to establish that the solution to this problem is unique.

(b) The general solution to the above problem for $u(x,t)$ is made up of a sum of all the normal modes, with the n^{th} normal mode being given by

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi ct}{\ell} - \phi_n\right)$$

where A_n and ϕ_n are constants. Using the above expressions for T and V, find the kinetic energy T_n and potential energy V_n in this normal mode. Hence establish that the total energy E_n in each mode is constant throughout the motion of the string.

END