MATH261001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH2610**.

Only approved basic scientific calculators may be used

Examination for the Module MATH2610
(January 2006)

Oscillations and Waves

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. (a) Write down the equation (Euler-Lagrange equation) satisfied by the function $y(x)$, which takes the prescribed values $y(x_1) = a$, $y(x_2) = b$ and which makes the integral

$$I = \int_{x_1}^{x_2} f(x, y, p) dx$$

stationary, where $p \equiv \frac{dy}{dx}$. Deduce that, if f does not depend explicitly on x, then

$$p\frac{\partial f}{\partial p} - f = \text{constant}.$$

(b) Show that the function $y(x)$ which makes the integral

$$I = \int_{0}^{\pi/4} \left(4y^2 - p^2\right) dx$$

stationary, subject to $y(0) = 0$, $y(\frac{\pi}{4}) = 1$, satisfies the equation

$$\left(\frac{dy}{dx}\right)^2 = c - 4y^2$$

for some constant c.

Solve this equation, subject to the boundary conditions, to find the function $y(x)$. Hence show that the stationary value of I, is $I = 0$.

2. (a) The kinetic energy T an autonomous mechanical system specified by the n independent generalised co-ordinates $q_1, q_2, \cdots, q_n$ takes the form

$$T = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\, \dot{q}_i \dot{q}_j$$

where the a_{ij} depend only on the q_i $(i = 1, 2, \cdots, n)$ and $a_{ij} = a_{ji}$.
Show that

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T.$$

Use this result to show that the total energy $T + V$ is a constant, where $V = V(q_i)$ is the potential energy.

(b) The motion of a particle of mass m on the surface of a sphere of radius a can be described by the Lagrangian

$$L = \frac{1}{2} m a^2 \left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \right) - m g a \cos\theta,$$

where θ and ϕ are generalised co-ordinates and where g is the gravitational acceleration.

Show that $\sin^2\theta\, \dot{\phi}$ is a constant and use this result to obtain an equation of motion entirely in terms of θ (and its derivatives).

If the system starts at $t = 0$ with $\theta = \dfrac{\pi}{4}$, $\dot{\theta} = 0$, $\phi = 0$, $\dot{\phi} = 2$, show that this equation can be integrated to give

$$\frac{1}{2}\dot{\theta}^2 + \frac{1}{\sin^2\theta} + \frac{g}{a}\cos\theta = 2 + \frac{g}{a\sqrt{2}}.$$

3. A mechanical system, performing small oscillations about a position of stable equilibrium $x = y = 0$, has the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = (\frac{1}{2}\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2) - (\frac{1}{2}x^2 + xy + \frac{5}{2}y^2)$$

where x and y are generalised co-ordinates. Show that the normal frequencies of this system are given by $\omega_1^2 = 1$, $\omega_2^2 = 4$.

If, at $t = 0$,

$$x = 1 \quad y = 2, \quad \dot{x} = 0, \quad \dot{y} = 0$$

show that the subsequent motion is given by

$$x(t) = 3\cos t - 2\cos 2t, \qquad y(t) = 2\cos 2t.$$

4. Show that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on $0 < x < \ell$, $t > 0$, subject to the boundary conditions

$$u = 0, \quad \text{at } x = 0 \text{ and } x = \ell \quad (t > 0),$$

can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right).$$

Find the constants A_n and B_n in the case when initially $u = 0$ and

$$\frac{\partial u}{\partial t} = \begin{cases} v_0 & (0 < x < \frac{\ell}{2}), \\ 0 & (\frac{\ell}{2} < x < \ell). \end{cases}$$

5. (a) Show that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on $-\infty < x < \infty$, $t > 0$, can be expressed in the form

$$u(x,t) = f(x - ct) + g(x + ct).$$

Show that, if

$$u(x,0) = F(x), \quad \frac{\partial u}{\partial t}(x,0) = G(x), \quad (-\infty < x < \infty) \quad \text{at } t = 0,$$

$$u(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds.$$

(b) An infinite stretched string, initially at rest, is given the an initial displacement

$$u(x,0) = \begin{cases} x & (0 < x \le \frac{1}{2}) \\ 1 - x & (\frac{1}{2} \le x < 1) \\ 0 & \text{otherwise.} \end{cases}$$

Determine the subsequent motion and plot the solution at times $t = \frac{1}{4c}$, $t = \frac{1}{2c}$ and $t = \frac{1}{c}$.

END