MATH261001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH2610**.

Only approved basic scientific calculators may be used

Examination for the Module MATH2610
(January 2005)

Oscillations and Waves

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. Write down the equation (Euler-Lagrange equation) satisfied by the function $y(x)$, which takes the prescribed values $y(x_1) = a$, $y(x_2) = b$ and which makes the integral

$$I = \int_{x_1}^{x_2} f(x, y, p) dx$$

stationary, where $p \equiv \dfrac{dy}{dx}$. Deduce that, if f does not depend explicitly on x, then

$$p\frac{\partial f}{\partial p} - f = \text{constant}.$$

Show that the function $y(x)$ which makes the integral

$$I = \int_0^{\pi} \left(p^2 + \sin^2 y\right)^{1/2} dx$$

stationary, subject to $y(0) = \dfrac{\pi}{4}$, $y(\dfrac{\pi}{2}) = \dfrac{\pi}{2}$, satisfies the equation

$$\left(\frac{dy}{dx}\right)^2 = C\sin^4 y - \sin^2 y$$

for some constant C. Show that the substitution $u(x) = \cot y$ transforms this equation to

$$\left(\frac{du}{dx}\right)^2 = C - 1 - u^2.$$

Hence show that, if $C > 1$, the function y satisfies

$$\cot y = A\cos x + B\sin x, \qquad A, B \text{ constants}.$$

Determine the constants A and B from the boundary conditions.

2. (a) For an autonomous system with Lagrangian $L = L(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n)$, show that the quantity

$$E = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

is a constant.

(b) A mechanical system has the Lagrangian

$$L = \frac{1}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) - \frac{\mu}{(1-\cos\theta)}.$$

where θ and ϕ are generalised co-ordinates and where μ is a constant. Obtain two constants of the motion and deduce that

$$\frac{1}{2}\left(\dot{\theta}^2 + \frac{h^2}{\sin^2\theta}\right) + \frac{\mu}{(1-\cos\theta)} = E,$$

where h and E are constants. Find the value of these constants if, at $t = 0$,

$$\theta = \frac{\pi}{3}, \quad \dot{\theta} = 0, \quad \phi = 0, \quad \dot{\phi} = 1.$$

3. A mechanical system, performing small oscillations about a position of stable equilibrium $x = y = 0$, has the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = (\dot{x}^2 + \dot{y}^2) - (5x^2 - 4xy + 2y^2)$$

where x and y are generalised co-ordinates. Find the normal oscillatory frequencies of this system.

If, at $t = 0$,

$$x = 1, \quad y = 3, \quad \dot{x} = 0, \quad \dot{y} = 0$$

show that the subsequent motion is given by

$$x = \frac{1}{5}\left(7\cos t - 2\cos\left(\sqrt{6}t\right)\right), \qquad y = \frac{1}{5}\left(14\cos t + \cos\left(\sqrt{6}t\right)\right).$$

4. Show that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on $0 < x < \ell$, $t > 0$, subject to the boundary conditions

$$u = 0, \quad \text{at } x = 0 \text{ and } x = \ell \quad (t > 0),$$

can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right).$$

QUESTION 4 CONTINUED...

Indicate how the constants A_n and B_n are determined if u also satisfies the initial conditions

$$u = f(x), \qquad \frac{\partial u}{\partial t} = g(x) \qquad \text{at } t = 0 \quad (0 < x < \ell).$$

Find the constants A_n and B_n in the case when initially $\dfrac{\partial u}{\partial t} = 0$ and

$$u = \frac{u_0}{\ell^2} x(\ell - x), \qquad (0 < x < \ell)$$

where u_0 is a constant.

5. Show that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on $-\infty < x < \infty$, $t > 0$, can be expressed in the form

$$u(x,t) = f(x - ct) + g(x + ct).$$

Show that, if

$$u(x,0) = F(x), \quad \frac{\partial u}{\partial t}(x,0) = G(x), \quad (-\infty < x < \infty) \quad \text{at } \ t = 0,$$

$$u(x,t) = \frac{1}{2}\left(F(x - ct) + F(x + ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

An infinite stretched string, initially at rest, is given the an initial displacement

$$u(x,0) = \begin{cases} x(1-x) & (0 < x < 1) \\ 0 & \text{otherwise} \end{cases}$$

Determine the subsequent motion and plot the solution at times $t = \dfrac{1}{4c}$, $t = \dfrac{1}{2c}$ and $t = \dfrac{1}{c}$.

END