MATH260001

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH260001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2600
(January 2008)

Numerical Analysis

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. We shall look for approximations to the unique root of the function

$$f(x) = \cos(x) - 1, \quad x \in [-\pi/2, \pi/2].$$

(a) Find the value of the root of f in $[-\pi/2, \pi/2]$ and write down the mapping function $g(x)$ of the iterative map $p_{n+1} = g(p_n)$ for Newton's method.
Starting with an initial guess of $p_0 = 1.5$, calculate the next four approximations to the solution $f(x) = 0$ using Newton's method. For each iteration calculate the ratio $|p_n/p_{n-1}|$; what do these values suggest?

(b) Expand $g(p_n)$ as a Taylor series of in powers of p_n and find the convergence rate of the iteration.
Hint: you may want to use the approximations

$$\sin x = x + O(x^3) \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2} + O(x^4).$$

(c) In order to improve the convergence rate of Newton's method for solving $f(x) = 0$, let us define the function

$$h(x) = \frac{f(x)}{f'(x)}.$$

Write down the general step for solving $h(x) = 0$ using Newton's method and, taking $p_0 = 1.5$, calculate the next three approximations. For each iteration evaluate the ratio $|p_n/p_{n-1}^3|$.

(d) By means of a third order Taylor expansion, determine the convergence rate of the improved Newton's algorithm.

2. (a) Define $L_i(x)$, the i^{th} Lagrange polynomial of degree $(n-1)$, for n distinct points $x_1, x_2, \ldots, x_n$, and give the general expression of the Lagrange interpolation polynomial, for a function $f(x)$.

The value of a function f is known at the points $x = 1, 3$ and 4 and is given by

$$f(1) = 0, \quad f(3) = 4, \quad f(4) = 9. \tag{1}$$

Find the quadratic polynomial $P(x)$ that interpolates $f(x)$ through these points and simplify its expression.

(b) A function $S(x)$ is defined on the interval $[1, 4]$ by

$$S(x) = \begin{cases} S_1(x) = x^2 - 2x + 1, & x \in [1, 3), \\ S_2(x) = a(x-3)^2 + b(x-3) + c, & x \in [3, 4]. \end{cases}$$

Determine the constants a, b and c if S is a *quadratic* spline through the points in equation (1) — i.e. S interpolates f and is continuous with a continuous first derivative at the inner point.

(c) Let

$$P(x) = \sum_{i=1}^{n} x_i^m L_i(x),$$

for some positive integer $m \le (n-1)$ and, using properties of Lagrange polynomials, show that $P(x_k) = x_k^m$. Show further, from uniqueness of Lagrange interpolation, that $P(x) = x^m$.

(d) Let $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. Without calculation, give the expression of the cubic polynomial $P(x)$ that interpolates f through four points $x_1 < x_2 < x_3 < x_4$. Check whether $P(x)$ satisfies all the conditions to be the *natural cubic* spline that interpolates $f(x)$ through the same points.

3. (a) The trapezium rule for integrating a function over the interval $[0, h]$ can be written in the form,

$$\int_0^h f(x)\,\mathrm{d}x \approx w_0 f(0) + w_h f(h) \tag{2}$$

where w_0 and w_h are the weights. Use the method of undetermined coefficients to find the constants w_0 and w_h. Then, using a suitable quadratic polynomial, obtain the error term in the form $Kf''(\xi)$ for some value $\xi \in (0, h)$ where K is a constant to be determined.

(b) We shall construct the composite trapezium rule, including the error term, for approximating the definite integral

$$\int_a^b f(x)\mathrm{d}x.$$

Break the interval $[a, b]$ into n subintervals of constant width h and use the trapezium rule (2) on each subinterval to derive the composite trapezium rule.

Apply the Intermediate Value Theorem to the function $nf''(x)$ on (a,b) to show that the error term is of the form

$$E = -h^2\,\frac{b-a}{12}\,f''(\xi) \text{ for some } \xi \in (a,b). \tag{3}$$

(c) Determine from equation (3) the value of h required to approximate

$$I = \int_0^1 \mathrm{e}^x\,\mathrm{d}x$$

to within 10^{-1}, using the composite trapezium rule. Show that $n = 2$ subintervals are required to achieve this accuracy and calculate an approximation of the integral I using $n = 2$ in the composite trapezium rule. Show that the error in the approximation is consistent with the required accuracy.

4. (a) Let $Y(t)$ be the exact solution of the first order ordinary differential equation

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = f(t,Y). \tag{4}$$

By integrating equation (4) over the interval $[t_n, t_{n+1}]$ and approximating $f(t,Y)$ as a constant derive the forward Euler approximation,

$$y_{n+1} = y_n + hf(t_n, y_n),$$

where $h = t_{k+1} - t_k$ (for all k).

(b) The exact solution of the initial value problem

$$\frac{dY}{dt} = 2t, \qquad Y(0) = 0,$$

is $Y(t) = t^2$. Calculate the three first steps of the forward Euler method using a step-size of $h = 1/2$. At each step calculate the absolute error from the exact solution.

(c) Define the *local truncation error* of the numerical scheme and obtain its expression by means of a Taylor expansion of $Y(t_{n+1})$ in powers of h. Combine this Taylor expansion and the forward Euler approximation to show that the global error of the method is

$$E_{n+1} = E_n\left[1 + h\frac{\partial f(t_n, z_n)}{\partial z}\right] + \frac{h^2}{2}Y''(\xi_{n+1}), \tag{5}$$

where $E_n = Y(t_n) - y_n$.

Hints: The Mean Value Theorem implies that there exists z_n between $Y(t_n)$ and y_n such that

$$f(t_n, Y(t_n)) - f(t_n, y_n) = \frac{\partial f(t_n, z_n)}{\partial z}(Y(t_n) - y_n).$$

Check the formula (5) against the errors calculated in question 4(b).

5. (a) Explain what is meant by an LU-factorisation of a square matrix. Then, showing your method of working, find the LU-factorisation of the matrix,

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix},$$

and calculate its determinant. Demonstrate how this factorisation may be used to solve the linear system,

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 5 \end{pmatrix}.$$

(b) Show that the matrix,

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

cannot be factored into the product LU. Find the permutation matrix P, such that $M = PA$ can be factored. Find the LU-factorisation of M.

END