MATH260001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH260001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2600

(January 2005)

Numerical Analysis

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. The function $f(x) = x^2 - x - 1$ has a zero in the interval $[1, 2]$, equal to $(1+\sqrt{5})/2$.

(a) Describe briefly how this solution may be found using the bisection method, and perform the first three iterations. How many iterations are needed to guarantee that the absolute error is less than 10^{-4}?

(b) Write down the general step of Newton's method for solving the equation $f(x) = 0$, taking $p_0 = 1.5$ as an initial guess calculate the next three iterates. Find the absolute error in each iterate (given that $(1+\sqrt{5})/2 = 1.618033988$).

By treating Newton's method as an iterative map, show that as $n \to \infty$, the absolute error, e_n, in the nth iterate satisfies

$$e_n = \frac{1}{\sqrt{5}} e_{n-1}^2.$$

2. (a) For points $x_1 < x_2 < \cdots < x_n$, write down an expression for the polynomial $q_k(x)$ of degree $n-1$ that satisfies the conditions,

$$q_k(x) = \begin{cases} 1, & x = x_k; \\ 0, & x = x_i \qquad i \neq k \end{cases}$$

Hence find the equation of the polynomial $P(x)$, of degree $n-1$, such that

$$P(x) = f(x) \qquad \text{for } x = x_1, x_2 \ldots x_n.$$

Show that this polynomial is unique.

(b) Find the equation of the polynomial that interpolates $\cos(\pi x)$ through the points $x = 0, \frac{1}{3}$ and $\frac{1}{2}$. Use this polynomial to estimate the value of $\cos(\pi/6)$.
Using the error formula for Lagrange interpolation,

$$f(x) = P(x) + \frac{1}{n!}\prod_{i=1}^{n}(x - x_i)\frac{d^n f}{dx^n}(\xi), \qquad \text{for some } \xi \in (x_1, x_n),$$

find an upper bound for the error in your estimate above. How does this compare with the actual error?

3. (a) The Trapezium Rule for integrating a function over the interval [0, h] can be written in the form,

$$\int_0^h f(x)dx = w_0 f(0) + w_1 f(h) + K f''(\xi)$$

where w_0 and w_1 are constant weights and the final term on the right hand side is the error term with $\xi \in (0, h)$. Use the method of undetermined coefficients to find the constants w_0, w_1 and K.
Hence, neglecting the error term, derive the composite trapezium rule for approximating the integral

$$\int_a^b f(x)dx.$$

over four equal subintervals of $[a, b]$ and use this formula to estimate,

$$I = \int_0^1 \frac{1}{1+x}dx.$$

Calculate the absolute error from the true answer $I = \log(2) = 0.693147$

(b) Derive the three point open Newton-Coates formula for the interval $[-2h, 2h]$, in the form

$$\int_{-2h}^{2h} f(x)dx \simeq w_{-1}f(-h) + w_0 f(0) + w_1 f(h).$$

Use this quadrature scheme to evaluate the intergral I of part (a).

4. (a) Consider the ordinary differential equation,

$$\frac{dy}{dt} = f(t, y). \tag{1}$$

By integrating equation (1) over the interval $[t_n, t_{n+1}]$ and approximating $f(t, y)$ as a constant derive the forward Euler approximation,

$$y_{n+1} = y_n + hf(t_n, y_n),$$

where $h = t_{k+1} - t_k$ (for all k). Define the *local truncation error* of a numerical scheme and obtain an expression for the local truncation error of the forward Euler method, and show that it is first order accurate.

Verify that $y(t) = (t+1)e^{-t}$ is the solution the initial value problem

$$\frac{dy}{dt} = e^{-t} - y, \qquad y(0) = 1. \tag{2}$$

Use the forward Euler method to calculate an approximation to $y(1.0)$ using a step-size of $h = 0.5$ and calculate the absolute error from the exact solution $y(1) = 0.73576$. ($e^{-0.5} = 0.60653$)

(b) By finding a suitable quadrature formula for integration of equation (1) over the interval $[t_n, t_{n+1}]$ derive the two-step Adams-Bashforth scheme,

$$y_{n+1} = y_n + \frac{h}{4}\big(3f(t_n, y_n) - f(t_{n-1}, y_{n-1})\big).$$

By using the exact solutions for y_1 find the value of y_2 for equation(2) using the two-step Adams-Bashforth scheme with $h = 0.5$.

5. (a) Find the LU-factorisation of the matrix,

$$\begin{pmatrix} 3 & -2 & 4 \\ 6 & -3 & 8 \\ 1 & -1 & 1 \end{pmatrix},$$

showing your method of working. Demonstrate how this factorisation may be used to solve the linear system,

$$\begin{pmatrix} 3 & -2 & 4 \\ 6 & -3 & 8 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

(b) Explain what is meant by (row) *pivoting* in the context of matrix factorisation. Show that the linear system,

$$\begin{pmatrix} 0 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

cannot be solved by LU-factorisation without pivoting eventhough the matrix is non-singular. Show how pivoting overcomes this problem and hence find the solution.

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