MATH245001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH2450**.

Only approved basic scientific calculators may be used

Examination for the Module MATH2450
(January 2008)

Mathematics for Geophysical Sciences 3

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. A and B are the 3×3 matrices

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ -3 & 1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 6 \\ 2 & 4 & -5 \end{pmatrix}$$

Calculate

(i) $2A + B$ and $A - 3B$.

(ii) the products AB and BA.

(iii) the transpose matrices A^T and B^T.

(iv) the traces $tr(A)$, $tr(B)$, $tr(AB)$ and $tr(BA)$.

(v) the determinants $det(A)$, $det(B)$ and $det(AB)$.
Verify for this case that $det(AB) = det(A)det(B)$.

2. **(a)** Find the inverse of the 3×3 matrix

$$C = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ -2 & 0 & -4 \end{pmatrix}$$

QUESTION 2 CONTINUED...

(b) Show, without directly expanding the determinant, that

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (b-a)(c-a)(c-b)(a+b+c)$$

3. **(a)** Find the eigenvalues and corresponding eigenvectors of the 2×2 matrix

$$D = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$$

Hence find a a transformation matrix P for which $P^{-1} D P = \Lambda$, where Λ is a diagonal matrix.

(b) Show that the matrix D satisfies its characteristic polynomial.

(c) Show that the 2×2 matrix

$$G = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is orthogonal. Verify that $det(G) = 1$ and show that the eigenvalues are

$$\lambda = \frac{1 \pm i}{\sqrt{2}}$$

4. **(a)** For $\phi = 2x^2y + 5xy^2 - 7y^2z^2$

find

(i) $\nabla\phi$

(ii) the directional derivative at $(1,1,1)$ in the direction $(1,2,-1)$.

(iii) Find the tangent plane to the surface

$$2x^2y + 5xy^2 - 7y^2z^2 = -4$$

at the point $(1,2,1)$.

(b) For $\mathbf{q} = (x^2z,\ -xy^2,\ xyz)$

find $\operatorname{div}\mathbf{q}$ and $\operatorname{curl}\mathbf{q}$ and verify that $\operatorname{div}(\operatorname{curl}\mathbf{q}) = 0$.

CONTINUED...

5. **(a)** For the scalar field $\phi = x^2y^2z$ and the vector field $\mathbf{p} = (xy,\ xyz,\ 2xz^2)$, verify that

(i) $\operatorname{div}(\phi\,\mathbf{p}) = \phi\operatorname{div}\mathbf{p} + \mathbf{p}.\nabla\phi$

(ii) $\operatorname{curl}(\phi\,\mathbf{p}) = \nabla\phi\times\mathbf{p} + \phi\operatorname{curl}\mathbf{p}$

(b) Show that the parabolic cylinder co-ordinates (u, v, w) given in terms of the cartesian co-ordinates (x, y, z) by

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = w$$

are orthogonal and find the corresponding $h_1,\ h_2,\ h_3$.

6. **(a)** Evaluate the line integral $\int_A^B \mathbf{q}.d\mathbf{r}$, where

$$\mathbf{q} = (xy,\ 2xy,\ z^2)$$

around the positive quadrant of the circle $x^2 + y^2 = a^2$ given by

$$x = a\cos t, \quad y = a\sin t, \quad z = 0, \qquad 0 \le t \le \frac{\pi}{2}$$

from the point $A = (a, 0, 0)$ to the point $B = (0, a, 0)$.

(b) Calculate the surface flux integral $\int_{\mathcal{S}} \mathbf{q}.\mathbf{n}\,dS$ and the volume integral $\int_{\mathcal{V}} \operatorname{div}\mathbf{q}\,dV$ where

$$\mathbf{q} = (x,\ y,\ 0)$$

where $\mathcal{V}$ is the sphere $x^2 + y^2 + z^2 = a^2$, given in spherical polar co-ordinates by

$$x = r\sin\theta\cos\lambda, \quad y = r\sin\theta\sin\lambda, \quad z = r\cos\theta, \qquad 0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \lambda \le 2\pi$$

and $\mathcal{S}$ is the surface of this sphere, with outward normal $\mathbf{n}$.

Hence verify for this example that the Divergence Theorem holds.

END