MATH243101

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH2431**.

Only approved basic scientific calculators may be used

Examination for the Module MATH2431
(May/June 2004)

Fourier Series, Partial Differential Equations and Transforms

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. Any piecewise smooth function $f(x)$ on $-\ell < x < \ell$ can be represented by the Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right].$$

Give the formulae for calculating the constants a_n and b_n for a given function f.

State what is meant by an *odd* function and show that, for an odd function, the a_n are all zero.

Determine the Fourier series for the function

$$f(x) = x \qquad \text{on} \quad -\ell < x < \ell.$$

Explain why your Fourier series has the value zero at $x = \ell$ even though $f(\ell) \neq 0$.

Use your Fourier series for $f(x)$ to establish the result that

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} = \frac{\pi}{4}.$$

2. It is required to solve the steady heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the square region $0 < x < a$, $0 < y < a$, subject to the boundary conditions

$$\begin{aligned} u &= 0 \quad \text{on } x = 0 \text{ and } x = a \quad (0 < y < a), \\ u &= 0 \quad \text{on } y = 0 \quad (0 < x < a), \\ u &= 1 \quad \text{on } y = a \quad (0 < x < a). \end{aligned}$$

Show, using the method of separation of variables, that the solution which satisfies the conditions on $x = 0$, $x = a$ and on $y = 0$ can be expressed as

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh(\frac{n\pi y}{a}) \sin(\frac{n\pi x}{a})$$

for constants B_n.

Show how the boundary condition on $y = a$ can be used to determine the B_n. Hence find the B_n.

3. Small transverse vibrations of a uniform string of line density ρ_0 and length ℓ, stretched to a tension T_0, are described by solutions $u(x,t)$ of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{(with } c = \sqrt{\frac{T_0}{\rho_0}}\text{)}$$

on $0 < x < \ell$, $t > 0$, subject to the boundary conditions

$$u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell \quad (t > 0),$$

Show that this boundary-value problem has the general solution

$$u(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{n\pi ct}{\ell}\right) + D_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right).$$

Explain how the constants C_n and D_n can be determined if u also satisfies the initial conditions

$$u = f(x), \qquad \frac{\partial u}{\partial t} = g(x) \qquad \text{at} \quad t = 0 \quad (0 < x < \ell).$$

Find the constants C_n and D_n in the case when the string is initially at rest $(g(x) = 0)$ and is given an initial displacement

$$u(x,0) = f(x) = \frac{u_0}{\ell^2}\, x(\ell - x).$$

4. Define the Fourier transform $F[f] = \overline{f}(\omega)$ of the function $f(x)$, and express the Fourier transform of $\dfrac{d^2 f}{dx^2}$ in terms of $F[f]$.

Show that the Fourier transform of

$$f(x) = \mathrm{e}^{-a|x|} \qquad \text{(where } a \text{ is a positive real constant)}$$

is

$$\overline{f}(\omega) = \frac{2a}{a^2 + \omega^2}.$$

 QUESTION 4 CONTINUED...

State the Inversion Formula for Fourier transforms.

Use Fourier transforms to solve the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on $-\infty < x < \infty$, $t \geq 0$, subject to the conditions

$$\begin{aligned} u &= \mathrm{e}^{-|x|} && \text{at } \ t = 0 \qquad (-\infty < x < \infty), \\ u &\to 0 && \text{as } \ |x| \to \infty \ \ (t > 0). \end{aligned}$$

By noting that the Fourier transform of the solution is an *even* function of ω, obtain the solution as a real integral with respect to ω.

5. (i) Write down an expression for $\bar{f}(p)$, the Laplace transform of a function $f(t)$ defined for $t > 0$.
Establish the results that $\mathcal{L}\left[\frac{df}{dt}\right] = p\bar{f}(p) - f(0)$ and $\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = p^2\bar{f}(p) - pf(0) - f'(0)$.

Find the Laplace transform of the function $f(t) = \mathrm{e}^{a\,t}$ (where a is a real constant).

Establish the result that $\mathcal{L}[tf(t)] = -\frac{d\bar{f}}{dp}$, and use this result to find the Laplace transform of $f(t) = t\,\mathrm{e}^{at}$.

(ii) $x(t)$ is defined to be the solution to the initial-value problem

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = \mathrm{e}^{2t} \qquad (t > 0),$$

$$x = 1, \quad \frac{dx}{dt} = 0 \qquad \text{at } \ t = 0.$$

Show that its Laplace transform is

$$\overline{x}(p) = \frac{p^2 - 4p + 5}{(p-1)^2(p-2)}.$$

Resolve this expression into partial fractions and hence find $x(t)$.

END