MATH243101

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH2431**.

Only approved basic scientific calculators may be used

Examination for the Module MATH2431
(May/June 2003)

Fourier Series, Partial Differential Equations and Transforms

Time allowed: **2 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. Find the Fourier series for the function

$$f(x) = x^2$$

on the interval $-\ell < x < \ell$.

Sketch the graph of the function represented by the series on the range $-3\ell < x < 3\ell$. By putting $x = 0$ in your series, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

2. Show that the general solution to the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on $0 < x < \ell$, $t > 0$, subject to the boundary conditions

$$u = 0 \qquad \text{at} \;\; x = 0 \text{ and } \; x = \ell \quad (t > 0),$$

is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \, \mathrm{e}^{-\frac{n^2\pi^2}{\ell^2}t} \sin\left(\frac{n\pi x}{\ell}\right).$$

Determine the constants B_n in the case when u also satisfies the initial condition

$$u = \frac{u_0}{\ell}(\ell - x) \quad \text{at } t = 0 \qquad (0 < x < \ell),$$

where u_0 is a constant.

3. Given that the general solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

on $0 < x < \ell$, $t > 0$, subject to the boundary conditions

$$u = 0, \quad \text{at } x = 0 \text{ and } x = \ell \quad (t > 0),$$

is

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right),$$

indicate how the constants A_n and B_n can be determined if u also satisfies the initial conditions

$$u = f(x), \qquad \frac{\partial u}{\partial t} = g(x) \qquad \text{at } t = 0 \quad (0 < x < \ell).$$

Find the constants A_n and B_n in the case when initially $u = 0$ and

$$\frac{\partial u}{\partial t} = \begin{cases} 0 & (0 < x < \frac{\ell}{3}) \\ v_0 & (\frac{\ell}{3} < x < \frac{2\ell}{3}) \\ 0 & (\frac{2\ell}{3} < x < \ell), \end{cases}$$

where v_0 is a constant.
Given that u is the small, transverse displacement of a uniform stretched string of length ℓ and line density ρ_0, show that the total energy E in the oscillations is $E = \dfrac{\rho_0 V_0^2 \ell}{6}$.
Find the energy E_1 in the first mode and show that this represents approximately 61% of the total energy.

4. Define the Fourier transform $F[f]$ of the function $f(x)$. Use this result to express the Fourier transform of $\dfrac{d^2 f}{dx^2}$ in terms of $F[f]$.

Show that the Fourier transform of

$$f(x) = e^{-ax^2} \qquad \text{(where } a \text{ is a positive real constant)}$$

is

$$\overline{f}(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\omega^2/4a}.$$

State the Convolution Theorem for Fourier transforms.

Use Fourier transforms and the Convolution Theorem to solve the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

 QUESTION 4 CONTINUED...

on $-\infty < x < \infty$, $t \geq 0$, subject to the conditions

$$\begin{aligned} u &= f(x) \qquad \text{at } t = 0 \quad (-\infty < x < \infty), \\ u &\to 0 \qquad \text{as } |x| \to \infty \ (t > 0). \end{aligned}$$

You should leave your result as an integral.

5. (i) Write down an expression for $\bar{f}(p)$, the Laplace transform of a function $f(t)$ defined for $t > 0$.

Establish the result that $\mathcal{L}\left[\dfrac{df}{dt}\right] = p\bar{f}(p) - f(0)$.

Find the Laplace transform of the function $f(t) = \mathrm{e}^{a\,t}$ (where a is a real constant).

(ii) Use Laplace transforms to find $x(t)$, where $x(t)$, $y(t)$ satisfy the initial-value problem

$$\begin{aligned} \dot{x} &= \frac{7}{3}x + \frac{8}{3}y, \\ \dot{y} &= \frac{1}{3}x + \frac{5}{3}y, \end{aligned}$$

subject to the initial conditions that

$$x(0) = 3, \quad y(0) = 0.$$

END